ON CERTAIN MATRICES AND THEIR DETERMINANTS*

BY W. E. ROTH

1. *Introduction.* It is well known that certain determinants, $|c_{ij}|$, $(i, j=1, 2, \cdots, r)$, are expressible as the products of r linear factors of the form $a_{i1}\omega_{i1}+a_{i2}\omega_{i2}+\cdots+a_{ir}\omega_{ir}$, $(i=1, 2, \cdots, r)$, where a_{ij} are rational functions of the elements, c_{1j}, in the first row of $|c_{ij}|$, and where ω_{ij} are functions of certain roots of unity and depend only on the relations existing between the elements of succeeding rows and those of the first row of the given determinants. Cyclic determinants and some related types which have this property are well known.†

In the present paper a generalization of these results is carried forward in two directions; first, we replace the scalar elements in the array of the given determinant by square matrices of order n; and second, we permit succeeding rows of the array to be other than permutations of the matrices occurring say in the first row of the given determinant. Certain special types, which we here generalize, have been studied by Puchta, Noether, Baltzer, Drude, Burnside, Scorza, and others.‡

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† Pascal, *Die Determinanten*, §§20–21. Gegenbauer, *Über eine specielle symmetrische Determinante*, Sitzungsberichte, Akademie der Wissenschaften, Mathematische-Naturwissenschaftliche Klasse, vol. 82, II–III (1880), pp. 938–942. Burnside, *On a property of certain determinants*, Messenger of Mathematics, vol. 23 (1894), pp. 112–114.

‡ Puchta, *Ein Determinantensatz und seine Umkehrung*, Denkschriften der Wiener Akademie, vol. 38, 2te Abth. (1878), pp. 215–221; *Ein neuer Satz aus der Theorie der Determinanten*, ibid., vol. 44, 2te Abth. (1882), pp. 227–282.

Noether, *Zur Theorie der Thetafunctionen*, Mathematische Annalen, vol. 16 (1880), pp. 322–325; *Notiz über eine Classe symmetrischer Determinanten*, ibid., pp. 551–555.

Baltzer, *Ueber einen Satz aus der Determinantentheorie*, Nachrichten, Königlich Gesellschaft der Wissenschaften, Göttingen, 1887, pp. 389–391.

Drude, *Ein Satz aus der Determinantentheorie*, ibid., pp. 118–122.

Burnside, loc. cit.

Scorza, *Sopra una certa classe di determinanti e sulla forma Hermitiane*, Giornale di Matematica, vol. 51 (1913), pp. 335–342.

2. *Direct Product Matrices*. If A and B are $n\times n$ matrices and if

$$U = (u_{ij}), \qquad V = (v_{ij}), \qquad (i, j = 1, 2, \cdots, r),$$

are $r\times r$ matrices whose elements u_{ij} and v_{ij} are commutative with those of A and of B, then the $nr\times nr$ matrix

$$\langle A\rangle U = U\langle A\rangle = (u_{ij}A) = (Au_{ij}),$$

is the direct product of A and U, and it readily follows that

$$\langle A\rangle U\cdot\langle B\rangle V = \langle AB\rangle(UV) = (UV)\langle AB\rangle. \tag{1}$$

Moreover, the matrices $\langle A\rangle U$ and $A\langle U\rangle$ are equivalent in that their characteristic matrices have identical elementary divisors.*

3. *The Matrix* $\langle A_0\rangle I_r+\langle A_1\rangle U+\cdots+\langle A_{r-1}\rangle U^{r-1}$. We shall assume that the elements of the $n\times n$ matrices, A_i, $(i=0, 1, \cdots, r-1)$, belong to the field F, and that the $r\times r$ matrix U, likewise with elements in F, is such that $U-\lambda I$ has the elementary divisors $(\theta_i-\lambda)^{r_i}$, $(i=1, 2, \cdots, s)$, where θ_i, $(i=1, 2, \cdots, r)$, are not necessarily in F. Let the non-singular matrix, Q, with elements in F extended by the adjunction of the roots of $|U-\lambda I|=\phi(\lambda)=0$, be such that

$$QUQ^{-1} = U_1 \dotplus U_2 \dotplus \cdots \dotplus U_s, \tag{2}$$

where

$$U_i = \begin{pmatrix} \theta_i & 1 & 0\cdots 0 \\ 0 & \theta_i & 1\cdots 0 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot \\ 0 & 0 & 0\cdots 1 \\ 0 & 0 & 0\cdots \theta_i \end{pmatrix} \tag{3}$$

is an $r_i\times r_i$ matrix and $\sum_{i=1}^{s} r_i=r$. For convenience, we shall designate as $B(U)$ the $nr\times nr$ matrix discussed in this section:

$$B(U) = \langle A_0\rangle I_r + \langle A_1\rangle U + \cdots + \langle A_{r-1}\rangle U^{r-1},$$

where I_r is the unit matrix of order r. Hence according to (1) and (2) we have

* For further details on the algebra of direct product matrices see MacDuffee, *The Theory of Matrices*, Springer, 1933, Chapter VII; and Roth, *On direct product matrices*, this Bulletin, vol. 40 (1934), pp. 461–468.

$$\langle I_r\rangle Q\cdot B(U)\cdot\langle I_r\rangle Q^{-1} = B(U_1) \dotplus B(U_2) \dotplus \cdots \dotplus B(U_s),$$

and

$$(4)\qquad B(U_i) = \begin{Bmatrix} B(\theta_i) & B'(\theta_i) & \cdots & B^{(r_i-1)}(\theta_i) \\ 0 & B(\theta_i) & \cdots & B^{(r_i-2)}(\theta_i) \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & B(\theta_i) \end{Bmatrix},$$

where

$$B^{(k)}(\theta_i) = C_{k,k}A_k + C_{k+1,k}A_{k+1}\theta_i + \cdots + C_{r_i-1,k}A_{r_i-1}\theta^{r_i-k-1},$$

$(k=0, 1, \cdots, r_i-1;\ i=1, 2, \cdots, s)$, the C's being binomial coefficients. We have proved the theorem:

THEOREM. *If*

$$B(U) = \langle A_0\rangle I_r + \langle A_1\rangle U + \cdots + \langle A_{r-1}\rangle U^{r-1},$$

where the A_i are $n\times n$ matrices with elements in F, and if U, an $r\times r$ matrix with elements in F, is equivalent to the direct sum $U_1\dotplus U_2\dotplus \cdots \dotplus U_s$, where the U_i, $(i=1, 2, \cdots, s)$, are given by (3); *then $B(U)$ is equivalent to the direct sum*

$$B(U_1) \dotplus B(U_2) \dotplus \cdots \dotplus B(U_s),$$

where the $B(U_i)$, $(i=1, 2, \cdots, s)$, given by (4) *have their elements in F extended by the adjunction of the roots of $|U-\lambda I|=0$.*

This theorem, whose proof as given above is almost trivial, has very extensive applications in the theory of determinants. A few of these will be illustrated in the following section.

4. *Determinants.* The theorem above leads at once to the following:

COROLLARY. *If θ_i, $(i=1, 2, \cdots, r)$, are the characteristic roots of U, whether distinct or not, then*

$$(5)\qquad |B(U)| = \prod_{i=1}^{r} |B(\theta_i)|.*$$

* Williamson (*The latent roots of a matrix of special type*, this Bulletin, vol. 37 (1931), pp. 585–590, Theorem I) and Finan (*A theorem on matrices*, abstract 41-11-359, this Bulletin) have obtained results closely allied to this corollary.

That is, the $nr\times nr$ determinant $|B(U)|$ is expressible as the product of r determinants, $|B(\theta_i)|$, $(i=1, 2, \cdots, r)$, of order n. For example, the determinants studied by Baltzer* and by Scorza† would, according to our notation, be given by

$$\Delta = \begin{vmatrix} A_0 & A_1 \\ -A_1 & A_0 \end{vmatrix} = |\langle A_0\rangle I_2 + \langle A_1\rangle U|, \text{ where } U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and since $|U-\lambda I| = (i-\lambda)(-i-\lambda)$, we have

$$\Delta = |A_0 + iA_1| \cdot |A_0 - iA_1|.$$

Hence the given determinant is the sum of two squares provided the elements of A_0 and A_1 are real. Also we can conclude that if A_0+iA_1, where A_0 and A_1 are real, have a real characteristic root, then $\langle A_0\rangle I_2+\langle A_1\rangle U$ above has this root as a multiple characteristic root.‡

The cyclic determinant

$$\begin{vmatrix} A_0, & A_1, \cdots, A_{r-1} \\ A_{r-1}, & A_0, \cdots, A_{r-2} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_1, & A_2, \cdots, A_0 \end{vmatrix} = |B(U)|,$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ 0 & 0 & 0 \cdots 1 \\ 1 & 0 & 0 \cdots 0 \end{pmatrix}$$

and the A_i, $(i=0, 1, \cdots, r-1)$, are $n\times n$ matrices, is given by the product

$$\prod_{i=1}^{r} |A_0 + A_1\omega^i + \cdots + A_{r-1}\omega^{(r-1)i}|,$$

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* Baltzer, loc. cit.

† Scorza, loc. cit.

‡ Wedderburn, *Lectures on Matrices*, p. 101.

where ω is a primitive rth root of unity, that is, a primitive root of $|U-\lambda I_r|=0$.

Each of the factors, $|B(\theta_i)|$, $(i=1, 2, \cdots, r)$, in (5) is again expressible as the product of s factors, if the $n\times n$ matrices A_i, $(i=0, 1, \cdots, r-1)$, are given by

$$A_i = \langle A_{i,0}\rangle I_s + \langle A_{i,1}\rangle V + \cdots + \langle A_{i,s-1}\rangle V^{s-1},$$

where the A_{ij}, $(i=0, 1, \cdots, r-1; j=0, 1, \cdots, s-1)$, are $m\times m$ matrices, where V is an $s\times s$ matrix, and $n=ms$. This factorization may be continued still farther under certain conditions which are now obvious.

Matrices of the form given by $B(U)$ above have arisen frequently in the literature, particularly in the theory of algebraic numbers. For example, the $n\times n$ matrix whose elements are algebraic numbers with θ, a root of the equation

$$a_0 + a_1\lambda + \cdots + a_{r-1}\lambda^{r-1} - \lambda^r = 0,$$

as basis, can be written

$$B(\theta) = A_0 + A_1\theta + \cdots + A_{r-1}\theta^{r-1},$$

where the A_i, $(i=0, 1, \cdots, r-1)$, are $n\times n$ matrices. Then

$$\theta B(\theta) = A_0' + A_1'\theta + \cdots + A_{r-1}'\theta^{r-1},$$

where

$$A_0' = a_0A_{r-1}, \qquad A_i' = A_{i-1} + a_iA_{r-1}, \quad (i = 1, 2, \cdots, r-1).$$

Similarly

$$\theta^2 B(\theta) = A_0'' + A_1''\theta + \cdots + A_{r-1}''\theta^{r-1},$$

where

$$A_0'' = a_0A_{r-1}', \quad A_i'' = A_{i-1}' + a_iA_{r-1}', \quad (i = 1, 2, \cdots, r-1),$$

and so on. Hence the $nr\times n$ matrix

$$\begin{pmatrix} B(\theta) \\ \theta B(\theta) \\ \vdots \\ \theta^{r-1}B(\theta) \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ A_0' & A_1' & \cdots & A_{r-1}' \\ \cdot & \cdot & \cdots & \cdot \\ A_0^{(r-1)} & A_1^{(r-1)} & \cdots & A_{r-1}^{(r-1)} \end{pmatrix} \cdot \langle I_r\rangle \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{r-1} \end{pmatrix},$$

$$= B(U)\cdot\langle I_r\rangle \begin{Bmatrix} 1 \\ \theta \\ \cdot \\ \cdot \\ \cdot \\ \theta^{r-1} \end{Bmatrix},$$

where

$$(6) \qquad U = \begin{Bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{r-1} \end{Bmatrix}$$

and

$$B(U) = \langle A_0\rangle I_r + \langle A_1\rangle U + \cdots + \langle A_{r-1}\rangle U^{r-1}.$$

The $nr\times nr$ matrix $B(U)$ in F corresponds to the $n\times n$ matrix $B(\theta)$, whose elements are algebraic numbers, in the sense of an isomorphism under addition and multiplication.*

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* The determinant, $B(U)$, may be regarded as the norm of $B(\theta)$. See Dickson, *Algebras and their Arithmetics*, p. 70.