$$\{[f(x + (D - 1)\theta) - f(x - \theta D)]F(y)\}_{y=0} = \theta \frac{df(x)}{dx} \cdot$$

Blissard's remark, "*An equation which has a representative quantity is not susceptible to any algebraic operation by which the indices would be affected,*" becomes

$$(Df)^2 \neq D^2 f.$$

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ON FOURTH ORDER SELF-ADJOINT DIFFERENCE SYSTEMS*

BY V. V. LATSHAW

A linear difference expression for which the differential transform is self-adjoint (anti-self-adjoint) we shall call self-adjoint (anti-self-adjoint).† We choose two fourth order difference equations

$$(1)\quad \begin{aligned} L^{+}(u) \equiv\ & p(x)[u(x + 2) + u(x - 2)] \\ & + \lambda[u(x + 1) + u(x - 1)] + R(x)u(x) = 0, \end{aligned}$$

$$(2)\quad \begin{aligned} L^{-}(u) \equiv\ & p(x)[u(x + 2) - u(x - 2)] \\ & + \lambda[u(x + 1) - u(x - 1)] = 0, \end{aligned}$$

where $L^{+}(u)$ is self-adjoint and $L^{-}(u)$ anti-self-adjoint for the range $(x=a, a+1, \cdots, b-1; b-a \geqq 4)$. $R(x)$ and $p(x)$ are both real, $p(x)$ being a non-vanishing periodic function of period two; λ is a parameter.

Let the functions (y_1, y_2, y_3, y_4) constitute a fundamental set of solutions for either (1) or (2), and (w_1, w_2, w_3, w_4) the set adjoint to it. The two sets are related by the equations

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† J. Kaucky, *Sur les équations aux différences finies qui sont identiques à leurs adjointes*, Publications of the Faculty of Sciences, University of Masaryk, No. 22 (1922). For a discussion of adjoint differential expressions of infinite order, see H. T. Davis, *The Theory of Linear Operators*, 1936, pp. 474–475.

$$w_j(x)p(x) = \frac{A_{4j}(x+1)}{A(x+1)},$$

(3)
$$A(x) = \begin{vmatrix} y_1(x-2) & \cdots & y_4(x-2) \\ \cdot & \cdots & \cdot \\ y_1(x+1) & \cdots & y_4(x+1) \end{vmatrix},$$

in which $A(x)$ denotes the Casorati determinant and $A_{ij}(x)$ the cofactors of its elements. Since w_j satisfies the equation $L(u)=0$, one can write

(4) $$w_i = c_{i1}y_1(x) + \cdots + c_{i4}y_4(x), \qquad (i = 1, 2, 3, 4).$$

The c_{ij} have properties stated in the following theorem.*

THEOREM 1. *The matrix C^+ of the substitution* (4) *is skew-symmetric and C^- symmetric. Further, there is a set of relations involving c_{ij}^+ and the second order minors of $A^+(x)$ from which the c_{ij}^+ may be calculated explicitly*:

$$\sum_{j>i=1}^{4} c_{ij}^+\begin{pmatrix} y_i & y_j \\ -2 & -1\end{pmatrix}^- = 0, \qquad \sum c_{ij}^+\begin{pmatrix} y_i & y_j \\ -2 & 0\end{pmatrix}^- = -\frac{1}{p(x)},$$

$$\sum c_{ij}^+\begin{pmatrix} y_i & y_j \\ -2 & 1\end{pmatrix}^- = \frac{\lambda}{p(x)p(x+1)},$$

$$\sum_{j>i=1}^{4} c_{ij}^+\begin{pmatrix} y_i & y_j \\ -1 & 0\end{pmatrix}^- = 0, \quad \sum c_{ij}^+\begin{pmatrix} y_i & y_j \\ -1 & 1\end{pmatrix}^- = -\frac{1}{p(x+1)},$$

$$\sum c_{ij}^+\begin{pmatrix} y_i & y_j \\ 0 & 1\end{pmatrix}^- = 0.$$

We resolve (4) to get

(5) $$c_{ij}^+ = \left[w_i(x-2)A_{1j}(x) + \cdots + w_i(x+1)A_{4j}(x)\right]\frac{1}{A(x)}\cdot$$

We use (1, 2, 3) to furnish the following relations

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* We introduce a $(+)$ and $(-)$ convention to distinguish between quantities associated with (1) and (2) respectively. Also we set

$$u(x+r)v(x+s) \pm v(x+r)u(x+s) = \begin{pmatrix} u & v \\ r & s\end{pmatrix}^{\pm}.$$

$$
(6)\quad
\begin{aligned}
w_j(x-1)p(x-1) &= \frac{A_{4j}(x)}{A(x)},\\
w_j(x-2)p(x) + \lambda w_j(x-1) &= \frac{A_{3j}(x)}{A(x)},\\
\pm\, w_j^{\pm}(x)p(x) &= \frac{A_{1j}^{\pm}(x)}{A^{\pm}(x)},\\
\mp\, [\lambda w_j^{\pm}(x) + p(x+1)w_j^{\pm}(x+1)] &= \frac{A_{2j}^{\pm}(x)}{A^{\pm}(x)}.
\end{aligned}
$$

Combining (5) and (6) we find

$$
(7)\quad
\begin{aligned}
c_{ij}^{\pm} = p(x+1)\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 1 & -1 \end{pmatrix}^{\mp} &+ p(x)\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 0 & -2 \end{pmatrix}^{\mp} \\
&+ \lambda\begin{pmatrix} w_i^{\pm} & w_j^{\pm} \\ 0 & -1 \end{pmatrix}^{\mp}.
\end{aligned}
$$

Obviously we have $c_{ij}^{+} = -c_{ji}^{+}$ and $c_{ij}^{-} = c_{ji}^{-}$. Combine (3) and (4) to eliminate the w's. These equations together with $L^{+}(y_i^{+}) = 0$ yield the set of equations involving the minors. The sixth order determinant composed of the two-rowed minors is non-vanishing ($=[A^{+}(x)]^3$).*

In the development of adjoint difference systems the formula

$$
(8)\quad \sum f(x+1) = f(x) + \sum f(x)
$$

is used to provide a Lagrange relation

$$
(9)\quad \sum_{x=a}^{b-1} [vL(u) - uL(v)] = \Pi(u, v) = U_1U_8 + \cdots + U_8V_1,
$$

in which the U_i are eight linearly independent forms arbitrarily chosen:

$$
(10)\quad
\begin{aligned}
U_i = {} & a_{i1}u(a-2) + \cdots + a_{i4}u(a+1) \\
& + b_{i1}u(b-2) + \cdots + b_{i4}u(b+1).
\end{aligned}
$$

If the two systems

$$
(11)\quad
\begin{aligned}
L(u) &= 0, \qquad U_1 = U_2 = U_3 = U_4 = 0,\\
L(v) &= 0, \qquad V_1 = V_2 = V_3 = V_4 = 0,
\end{aligned}
$$

* Turnbull, *The Theory of Determinants, Matrices, and Invariants*, 1929, p. 87.

are equivalent, we shall call them self-adjoint difference systems.

THEOREM 2. *Given the fourth order difference systems composed of the equations*

$$L^{\pm}(u) = 0, \qquad U_i = 0, \qquad (i = 1, 2, 3, 4),$$

defined in (1), (2), (10), *let* $u(x)$ *and* $v(x)$ *be any pair of functions satisfying* $U_i = 0$; *then* $\Pi(u, v) \equiv 0$ *is a necessary and sufficient condition that the given system be self-adjoint.*

That the above condition is necessary needs no proof. Let (u_1, u_2, u_3, u_4) be four linearly independent functions satisfying $U_i = 0$ and u any linear combination of them. Through substitution the identity $\Pi(u, v) \equiv 0$ gives

$$(12) \qquad U_5(u_i)V_4(u) + \cdots + U_8(u_i)V_1(u) = 0, \quad (i = 1, 2, 3, 4).$$

Since the set $(U_1, U_2, \cdots, U_8)$ is linearly independent, it follows that the four systems of constants comprising the coefficients of the V's are linearly independent and we have $V_1(u) = V_2(u) = V_3(u) = V_4(u) = 0$. A similar argument shows that any function satisfying the given boundary conditions will also satisfy the adjoint boundary conditions.

We record some examples which fulfill the condition for self-adjointness.

$$(13) \qquad \begin{aligned} U_1^{\pm} &= a_{11}^{\pm}u(a-2) + a_{12}u(a-1) + p(a)u(a) \\ &\quad + b_{13}u(b) + b_{14}u(b+1) = 0, \\ U_2^{\pm} &= \pm\, a_{12}u(a-2) + a_{22}^{\pm}u(a-1) + \lambda u(a) \\ &\quad + p(a+1)u(a+1) + b_{24}u(b+1) = 0, \\ U_3^{\pm} &= \pm\, b_{13}u(a-2) \pm p(b)u(b-2) \pm \lambda u(b-1) \\ &\quad + b_{33}^{\pm}u(b) + b_{34}u(b+1) = 0, \\ U_4^{\pm} &= \pm\, b_{14}u(a-2) \pm b_{24}u(a-1) \pm p(b+1)u(b-1) \\ &\quad \pm b_{34}u(b) + b_{44}^{\pm}u(b+1) = 0, \end{aligned}$$

with the agreement that $a_{11}^{-} = a_{22}^{-} = b_{33}^{-} = b_{44}^{-} = 0$. For self-adjoint Sturmian boundary conditions we make the added restrictions, $b_{13} = b_{14} = b_{24} = 0$.

We now introduce a function $G^{\pm}(x, t)$ defined for $(a-1 < x < b)$,

$(a \leqq t < b)$, satisfying the given boundary conditions and for which $L^{\pm}[G^{\pm}(i, j)] = \delta_{ij}$.

THEOREM 3. *Let* $\lambda_1^{\pm}, \lambda_2^{\pm}, \cdots$ *be sets of characteristic values for the systems* $L^{\pm}(u) = 0$, $U_i = 0$ *defined in* (13). *There exists a* λ_k^+ *in the interval* $(\lambda_i^- \leqq \lambda_k^+ < \lambda_{i+1}^-)$ *provided*

$$G^-(b, b-1) - G^+(b, b-1) - \sum_{x=a}^{b-1} [G^-(x-1, x) + G^+(x-1, x)] \neq 0, \quad (\lambda_i^- \leqq \lambda \leqq \lambda_{i+1}^-).$$

If $(b-a)$ *is an odd integer, then every value of* λ^- *is a characteristic value.*

Let $D^{\pm}(\lambda) = 0$ be the characteristic equations for (13). By writing $D^{\pm}(\lambda)$ in determinant form one finds

$$\frac{dD^{\pm}}{d\lambda} = 2D^{\pm}\left[G^{\pm}(b, b-1) \pm \sum_{x=a}^{b-1} G^{\pm}(x-1, x)\right]. \tag{14}$$

This relation enables us to write

$$\frac{d}{d\lambda}\left(\frac{D^-}{D^+}\right) = \frac{2D^-}{D^+}\left[G^-(b, b-1) - G^+(b, b-1) - \sum_{x=a}^{b-1} \{G^-(x-1, x) + G^+(x-1, x)\}\right]. \tag{15}$$

Between the two real consecutive zeros λ_i^- and λ_{i+1}^- of $D^-(\lambda)$ either the bracketed expression or $D^+(\lambda)$ must vanish. By assumption the bracketed expression does not vanish.

The proof of the final statement in the theorem consists merely in noticing that $D^-(\lambda)$ for this case is a skew-symmetric determinant of odd order.

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