TOPOLOGICAL PROPERTIES OF DIFFERENTIABLE MANIFOLDS*

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I. INTRODUCTION

1. *The Problems* In many fields of work one is led to the consideration of n-dimensional spaces. A given dynamical system has a certain number of "degrees of freedom"; thus a rigid body, with one point fixed, has three. A line in euclidean space is determined by four "parameters." We therefore consider the positions of the rigid body, or the straight lines, as forming a space of three, or four, dimensions.† But when we try to determine the points of the space by assigning to each a set of three, or four, numbers, we are doomed to failure. This is possible for a small region of either space, but not for the whole space at once. The best we can do is to cover the space with such regions, define a coordinate system in each, and state how the coordinate systems are related in any two overlapping regions. They will be related in general by means of differentiable,‡ or analytic, transformations, with non-vanishing Jacobian. Any such space we shall call a differentiable, or analytic, manifold.

For a complete study of such spaces, we must know not only properties of euclidean n-space E^n, which we may apply in each coordinate system separately, but also properties which arise from the manifold being pieced together from a number of such systems. It is these latter properties, essentially topological in character, which form the subject of the present address.

Suppose we wish to study differential geometry in the n-dimensional manifold M^n. At each point p of M^n, the possible differentials (or "tangent" vectors) form an n-dimensional vec-

* An address delivered before the Society on September 9, 1937, in State College, by invitation of the Program Committee.

† The first space forms the group G^3 of rotations in 3-space; it is homeomorphic with projective 3-space P^3. The second space is homeomorphic with the total space (see §2) of the tangent vector space of the projective plane P^2, or of the 2-sphere S^2 if we use oriented lines. This is easily seen by considering together all parallel lines.

‡ Differentiable will always mean continuously differentiable.

tor space $V(p)$, the so-called tangent space at p. For topological considerations, it is sufficient to consider vectors of unit length,* or directions, which form a sphere $S(p)$ of dimension $n-1$. This set of spheres forms the *tangent sphere-space* of M^n. Most of our work will be on the problem, how do the spheres fit together over the whole manifold? Suppose M^n is imbedded in a higher dimensional manifold M^m (for instance, euclidean E^m). Then we may consider the normal unit vectors at each point, forming an $(m-n-1)$-sphere, and thus the *normal sphere-space*. The methods we use in Part II were discovered independently by E. Stiefel [**11**]† and myself [**15**]. Here we shall not attempt always to describe the most elegant methods, but lean rather to the intuitive side.

In the last part we give some fundamental results, due to de Rham,‡ on the theory of multiple integration in a manifold. If we wish to integrate over an r-dimensional subset, and no measure function is given, we integrate "differential forms," in other words, "alternating covariant tensors" of order r; we shall call these simply r-functions. It is necessary for various purposes to find what "exact" r-functions exist. This problem, in any small region, is rather trivial; again, we must consider the whole manifold to answer it. The methods we describe are somewhat different from those of de Rham.

2. *Sphere-Spaces.* A *sphere-space* $S(K)$ is defined as follows. Let S_0^ν be the unit ν-sphere§ in $E^{\nu+1}$. Let K be a complex. To each point p of K let there correspond a ν-sphere $S(p)$; if $p\neq q$, we assume that $S(p)$ and $S(q)$ have no common points. For each closed cell σ of K and each point p of σ, let $\xi_\sigma(p)$ be a (1-1)map of S_0^ν into $S(p)$; let $\xi_\sigma(p, q)$ be the point into which the point q of S_0^ν goes. If the cells σ and σ' have the common point p, then

* As every differentiable manifold may be imbedded in a euclidean space (see the bibliography, Whitney [**14**], Theorem 1), we may define in it a Riemannian metric and thus lengths of vectors, and so on.

† Numbers in square brackets refer to the bibliography at the end of this paper.

‡ See [**6**], [**7**], and [**8**]. The main theorem was suggested by E. Cartan, Comptes Rendus de l'Académie des Sciences, Paris, vol. 187 (1928), pp. 196–198.

§ That is, the set of points $\sum x_i^2 = 1$ in $E^{\nu+1}$. A 0-sphere is a pair of points, a 1-sphere is a circle, and a 2-sphere is a spherical surface.

$\xi_\sigma(p)$ and $\xi_{\sigma'}(p)$ are both defined. We assume that

$$(1) \qquad \overset{-1}{\xi_{\sigma'}}(p)\xi_\sigma(p), \quad \text{that is,} \quad \overset{-1}{\xi_{\sigma'}}(p, \xi_\sigma(p, q)),$$

which is a (1-1) map of S_0^ν into itself, is orthogonal,* and varies continuously with p. We call K the *base space*, and the set $\mathfrak{S}(K)$ of all points on all $S(p)$, the *total space*.† If K were a general point set, we would replace the cells σ by open sets covering K. We might replace the $S(p)$ by vector spaces $V(p)$ (in the obvious manner for tangent and normal spaces); however, the topological properties of these are reducible to those of sphere-spaces. For any sphere-space $S(K)$ and subset L of K, there is a corresponding sphere-space $S(L)$, the *part of* $S(K)$ *over* L. We shall call the map $\xi_\sigma(p)$, (p in σ), the *σ-coordinate system* in $\mathfrak{S}(\sigma)$. It determines (for a fixed orientation of S_0^ν) one of two possible orientations of each $S(p)$, (p in σ).

We find invariants which serve to distinguish between sphere-spaces with the same base space by studying the following questions. First, is it possible to choose, for each p in K, a point $\phi(p)$ of $S(p)$, so that $\phi(p)$ is continuous in K? Any such map we shall call a *projection* of K into $\mathfrak{S}(K)$. More generally, is it possible to find k projections $\phi_1(p), \cdots, \phi_k(p)$ of K into $\mathfrak{S}(K)$, so that these are orthogonal in each $S(p)$, $(1 \leq k \leq \nu+1)$? If so, we call $S(K)$ *k-simple*. If $S(K)$ is $(\nu+1)$-simple, we call it *simple*.

For tangent sphere-spaces, the existence of a projection is obviously equivalent to the existence of a continuous field of non-vanishing (tangent) vectors. The only closed orientable surface on which there is such a field is the torus.‡ (If we try, on the sphere, making all vectors point north or east for instance, this fails at the two poles; see §6.) Note that if we can find one such field on an orientable surface, we can at once find a second field of orthogonal vectors; at each point, looking along the first vector, we let the second vector point to the left. It follows that the tangent space is simple in this case.

* It is sufficient to assume that the map is linear. However, this general case is easily reducible to that given. Because of the assumption, we may define orthogonality on any $S(p)$.

† For a study of some total spaces, see Hotelling [**4**], Seifert [**9**] and [**10**], Stiefel [**11**], and Threlfall [**12**].

‡ For a study of this problem in manifolds, see Hopf [**3**]; also Alexandroff-Hopf, *Topologie* I, pp. 548–552, and Stiefel [**11**].

Suppose $S(K)$ is simple. Let $q_1=(1, 0, \cdots, 0), \cdots, q_{\nu+1}=(0, 0, \cdots, 1)$ be the unit points of S_0^ν. Set $\zeta(p, q_i)=\phi_i(p)$, $(i=1, \cdots, \nu+1)$. Then for each p in K, we can define $\zeta(p, q)$ for all other q in S_0^ν uniquely so that it is an orthogonal map of S_0^ν into $S(p)$. Clearly ζ is (1-1) and continuous. Hence *if* $S(K)$ *is simple, then* $\mathfrak{S}(K)$ *is the cartesian product of* K *and* S_0^ν.

If the tangent space to M^n is simple, then to each direction $\zeta(p, q)$ at p corresponds a direction $\zeta(p', q)$, (same q), at any other point p'. Thus *a parallelism may be defined throughout* M^n *if and only if the tangent sphere-space is simple.*

Suppose the normal space to M^n in M^m is simple. Then corresponding to each normal vector function $\phi_i(p)$ we may define a function $f_i(p)$ in M^m with $\phi_i(p)$ as its gradient at p, and vanishing in M^n. We may then define M^n as the set of points in M^m at which all the $f_i(p)$ vanish. Conversely, if it is possible to define M^m in this manner, with the gradients of the $f_i(p)$ linearly independent, then the normal space is simple.*

II. Invariants of sphere-spaces

3. *Elementary Properties of Complexes.*† Let ${}^r\partial_i^j$ be the incidence number of the cells σ_i^r and σ_j^{r-1}. Define the *boundary* and *coboundary* of the chain $A^r=\sum\alpha_i\sigma_i^r$ by

$$\partial A^r = \sum {}^r\partial_i^j \alpha_i\sigma_j^{r-1}, \qquad \delta A^r = \sum {}^{r+1}\partial_j^i \alpha_i\sigma_j^{r+1}. \tag{2}$$

Then A^r is a *cycle* or *cocycle* if $\partial A^r=0$ or $\delta A^r=0$; A^r is *homologous* to B^r, $A^r\sim B^r$, if A^r-B^r is a boundary; similarly, for *cohomologous*, for which we use the symbol $\backsim$. If we identify homologous r-cycles, we obtain the rth *homology group*; similarly for the rth *cohomology group*. Define *scalar products* of r-chains by

$$\left(\sum \alpha_i\sigma_i^r\right)\cdot\left(\sum \beta_i\sigma_i^r\right) = \sum \alpha_i\beta_i. \tag{3}$$

Note that $A^r\cdot\sigma_i^r$ is the coefficient of σ_i^r in the chain A^r. As $\partial\sigma_i^r\cdot\sigma_j^{r-1}$ and $\sigma_i^r\cdot\delta\sigma_j^{r-1}$ both equal the incidence number ${}^r\partial_i^j$,

* See Seifert [10], and Whitney, Annals of Mathematatcis, vol. 37 (1936), pp. 865–878.

† For further details, see for instance Whitney, *Matrices of integers and combinatorial topology*, Duke Mathematical Journal, vol. 3 (1937), pp. 35–45, and *On products in a complex*, Annals of Mathematics, vol. 39 (1938). We shall refer to these papers as MI and PC respectively.

we have, on equating, multiplying by coefficients, and summing,

$$\partial A^r \cdot B^{r-1} = A^r \cdot \delta B^{r-1}. \tag{4}$$

We shall use σ also for the set of points in the closed cell σ, and $\partial\sigma$ for its point set boundary. Let K^r be the r-dimensional part of K, the sum of all cells of dimension $\leqq r$.

4. *Orientability of Sphere-Spaces.* First note that the part of $S(K)$ over K^0, $S(K^0)$, is simple. We can define $\mathfrak{S}(K^0)$ as a product by setting $\zeta(a) = \xi_a(a)$ for each vertex a. We now ask, under what conditions is $S(K^1)$ simple? We attempt to extend $\zeta(p)$ through the 1-cells of K. Consider any 1-cell $\sigma = ab$. $S(a)$ and $S(b)$ are given orientations both by ζ and by ξ_σ. If the orientations given by ζ are both the same as, or both opposite to, those

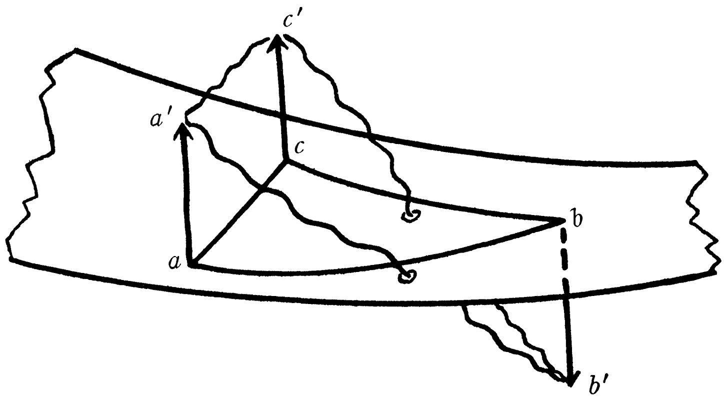

FIG. 1

given by ξ_σ, set $F^1(\sigma) = 0$; otherwise, $F^1(\sigma) = 1$. In Fig. 1,* if the end point of each vector, say at a, is $a' = \zeta(a, q_1)$, we have $F^1(ab) = F^1(bc) = 1$, $F^1(ca) = 0$. We may say, $F^1(\sigma)$ is 0 or 1 according as the orientations given to $S(a)$ and $S(b)$ by ζ are similar or opposite, when viewed through the σ-coordinate system. We define a *characteristic 1-cocycle* of K by letting $F^1(\sigma)$ be the coefficient of σ in a chain:

$$F^1 = \sum_i F^1(\sigma_i^1)\sigma_i^1, \quad \text{coefficients integers mod 2.} \tag{5}$$

* For p in ab, all vectors $\xi_{ab}(p)$ point up or all point down.

To show that F^1 is a cocycle, we shall find the coefficient $\delta F^1\cdot\sigma^2=F^1\cdot\partial\sigma^2$ of δF^1 in any 2-cell σ^2 of K. For each 1-cell ab of $\partial\sigma^2$, the similarity or dissimilarity of the orientations given to $S(a)$ and $S(b)$ when viewed through the ab-coordinate system is clearly the same as when viewed through the σ^2-coordinate system. Say $\partial\sigma^2=ab+bc+\cdots+ea$. If we compare the orientation of $S(a)$ with that of $S(b)$, then of $S(b)$ with that of $S(c),\ldots,$ the number of changes of orientation is even, as we come back to the original orientation of $S(a)$. Hence $F^1\cdot\partial\sigma^2\equiv 0 \pmod 2$, as required.

We might have chosen different coordinate systems over K^1, and have thus found a different F^1. To find the change made in F^1, suppose the orientation of $S(a)$ is changed,* while that of each other $S(x)$ is unaltered. Then for each 1-cell σ with a as a vertex, $F^1(\sigma)$ is changed from 0 to 1 or from 1 to 0, that is, it is increased by 1 (mod 2). Thus the new F^1 is the old plus δa (mod 2). Thus we may change F^1 only by coboundaries, and *the cohomology class $h^1(F^1)$ is an invariant of $S(K)$*. Clearly if F^1 is a characteristic cocycle, we may obtain any other $F'^1\backsim F^1$ by reorienting some of the $S(a)$ for vertices a.

We call the cohomology class $h^1=h^1(F^1)$ the *characteristic 1-class* of $S(K)$. If it vanishes, we call $S(K)$ *orientable*, otherwise *non-orientable*. Suppose $S(K)$ is orientable. Then we may reorient some of the $S(a)$ so that $F^1=0$. For each 1-cell ab, $S(a)$ and $S(b)$ now have either both the same or both the opposite orientations with the a- and b-coordinate systems as with the ab-coordinate system. In the latter case, change the orientation in the ab-coordinate system so that the orientations agree. Carrying out the same process in the 2-cells, and so on, we have finally chosen all coordinate systems so that for any point p of K in two cells σ and σ', their two coordinate systems give $S(p)$ the same orientation. This justifies the term "orientable."

To answer our original question, we shall show that *if $S(K)$ is orientable, and only then, it is simple over K^1*. If $h^1=0$, choose the ξ_a so that $F^1=0$ and set $\zeta(a)=\xi_a(a)$. Take any 1-cell ab. By hypothesis,

$$(6)\qquad \overset{-1}{\xi}_{ab}(a)\xi_a(a) \qquad\text{and}\qquad \overset{-1}{\xi}_{ab}(b)\xi_b(b)$$

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* That is, let Ω denote a reflection in S_0^ν, and set $\xi_a'(a)=\xi_a(a)\Omega$. Clearly F^1 is independent of the σ_i^1-coordinate systems.

are orthogonal transformations of S_0^ν into itself, that is, points $\psi(a)$ and $\psi(b)$ of the group $\bar{G}^{\nu+1}$ of orthogonal transformations in $(\nu+1)$-space. As $F^1(ab)\equiv 0$ (mod 2), $\psi(a)$ and $\psi(b)$ are in the same component of $\bar{G}^{\nu+1}$ (of which there are two), and hence we may join them by an arc. Let $\psi(p)$ run along this arc as p runs along ab, and set

$$\zeta(p) = \xi_{ab}(p)\psi(p) \tag{7}$$

in ab. Thus ζ is defined in ab, and in the same manner, throughout K^1. Conversely, if ζ is defined in K^1, set $\xi_a(a)=\zeta(a)$ for each vertex a; this makes $F^1=0$.

Let A be any 1-cycle (mod 2) of K; then $A\cdot F^1$ *is a numerical invariant* (mod 2) of $S(K)$. For if we had used F'^1 instead of F^1, then $F'^1-F^1=\delta G^0$ for some G^0, and

$$A\cdot F'^1 - A\cdot F^1 = A\cdot\delta G^0 = \partial A\cdot G^0 \equiv 0 \text{ (mod 2)}. \tag{8}$$

Any closed path in K defines a 1-cycle A; the orientation of $S(K)$ is "preserved" or "reversed" on going around the path according as $A\cdot F^1\equiv 0$ or 1 (mod 2).

It is easily seen that the tangent space to a manifold is orientable if and only if the manifold is orientable; the orientation of $S(K)$ is reversed on going around a path if and only if the orientation of the manifold is reversed. For an M^n in an M^m, the 1-class of the tangent space of M^n plus that of the normal space gives that of the part of the tangent space of M^m over M^n (mod 2).* For an example, consider a Möbius strip in E^3.

In the future, unless the contrary is specifically stated, we consider only orientable sphere-spaces.

5. *Higher Dimensional Invariants*. One can find, for oriented sphere-spaces, a characteristic cohomology class h^r for $2\leqq r \leqq \nu+1$, in the same manner as we defined orientability. We shall illustrate it with $\nu=1$, $r=2$; for instance, for the tangent space of S^2. As $S(K^1)$ is simple, we may define $\zeta(p, q)$ for p in K^1. Set $\phi(p)=\zeta(p, q_1)$; this is a projection of K^1 into $\mathfrak{S}(K^1)$. (For a general ν, we would use ν orthogonal projections in K^1.)

Now take any 2-cell σ of K, and consider the coordinate system ξ_σ in $\mathfrak{S}(\sigma)$. As p runs around $\partial\sigma$, $\psi(p)=\xi_\sigma(p, q_1)$ runs around a curve $C=\psi(\partial\sigma)$ on the torus $\mathfrak{S}(\partial\sigma)$ (see Fig. 2). Now consider

* Compare Seifert-Threlfall, *Topologie*, pp. 272–276.

$\phi(p)$. As p runs around $\partial\sigma$, it runs around a curve $D=\phi(\partial\sigma)$, cutting C in the positive sense (given by the orientation of $S(p)$) say α times; it is equivalent to running around C once, and α times around an $S(p)$. Set $F^2(\sigma)=\alpha$, and

$$(9) \qquad F^2 = \sum_i F^2(\sigma_i^2)\sigma_i^2, \quad \text{coefficients integers.}$$

This chain is easily seen to be independent of the ξ_{σ^2} (the orientation of the $S(p)$ is preserved). Just as before, we show that

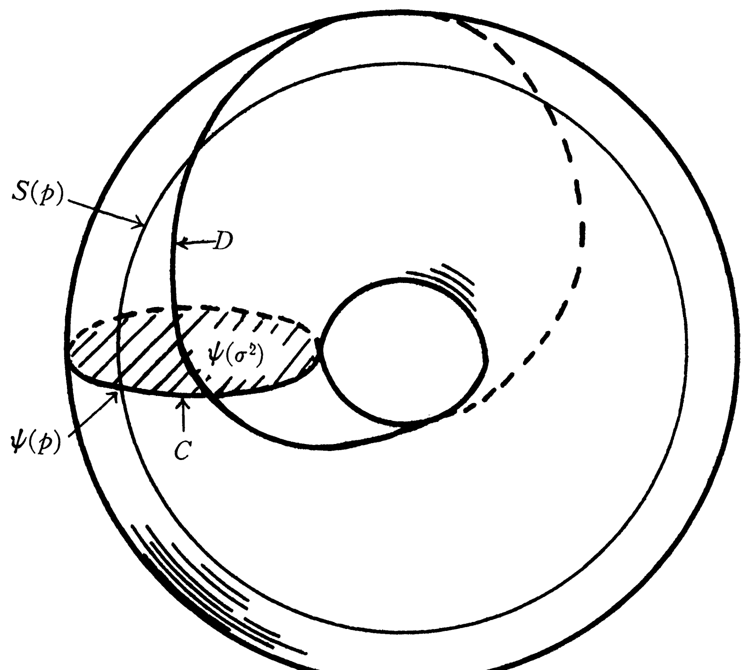

FIG. 2

it is a cocycle, and its cohomology class is an invariant of $S(K)$; also, we may obtain any $F'^2 \backsim F^2$.

For a general ν and r, we may always choose $\nu-r+2$ orthogonal projections of K^{r-1} into $\mathfrak{S}(K^{r-1})$; we then consider these maps in $\mathfrak{S}(\sigma^r)$, in relation to ξ_{σ^r}, to determine $F^r(\sigma^r)$ and thus F^r.

Suppose (see Stiefel [11], §6) we replace the first projection $\phi_1(p)$ used in defining F^r, for each p, by the diametrically opposite point of $S(p)$. It turns out that if r is odd, the new F'^r is $-F^r$. But $F'^r \backsim F^r$, hence $2F^r \backsim 0$, and $2h^r=0$. We find the general theorem:

For each r, $2 \leqq r \leqq \nu+1$, *there is a characteristic r-class* h^r, *which is an invariant of* $S(K)$; *the coefficients used are integers unless* $r<\nu+1$ *and r is even, in which case we reduce* mod 2. *If r is odd,* $2h^r=0$; $S(K)$ *is* $(\nu-r+2)$-*simple over* K^r *if and only if* $h^r=0$.

In regard to h^3, see §7.

For a non-orientable sphere-space, h^r is an invariant if we reduce mod 2 in all cases.*

As in §4, we can define a numerical invariant $A^r \cdot F^r$ corresponding to any cycle A^r. Further, if we do not reduce mod 2, then to any cycle A^r (mod μ), that is, such that $\partial A^r = \mu B^{r-1}$, corresponds the invariant (mod μ), $A^r \cdot F^r$; for if F'^r is another characteristic cocycle,

$$(10) \qquad A^r \cdot F'^r - A^r \cdot F^r = A^r \cdot \delta G^{r-1} = \mu B^{r-1} \cdot G^{r-1} \equiv 0 \pmod{\mu}.$$

We discuss the possibility of further invariants in Part II.

6. *Tangent and Normal Sphere-Spaces.* We shall consider (a) the highest dimensional invariant $F^{\nu+1}$ for tangent spaces, (b) the same for normal spaces, (c) relations between the spaces, and (d) other questions.

(a) In a closed orientable M^n, there is a "fundamental n-cycle" Z^n, whose multiples give all n-cycles. A single n-cell σ^n forms a cocycle, whose multiples $k\sigma^n$ determine all cohomology classes.† Any n-chain A^n is $\backsim k\sigma^n$ with $k=A^n \cdot Z^n$ = sum of coefficients of A^n. Hence, to find the characteristic n-class of $S(M^n)$, we need merely find $Z^n \cdot F^n$.

Let us do this for the tangent space of the 2-sphere S^2. The equator E cuts S^2 into the hemispheres σ_N and σ_S. Orient these so that $\partial\sigma_N = -\partial\sigma_S = E$; then $Z^2 = \sigma_N + \sigma_S$. To define F^2, we must choose a projection in $K^1 = E$, that is, a point $\phi(p)$ of $S(p)$ for each p in E. Let $\phi(p)$ be the direction at p pointing south. To find $F^2(\sigma_N)$, project σ_N stereographically onto the tangent plane

* Better, the theorem still holds if we replace chains in K by chains in an abstract complex K^* obtained as follows. The cells of K^* are those of K. Set ${}^r\partial_i'^j = \pm {}^r\partial_i^j$ (if the latter is $\neq 0$) according as the σ_i^r- and σ_j^{r-1}-coordinate systems determine the same or opposite orientations of $S(p)$ for p in σ_j^{r-1}.

† Compare MI, Appendix I.

P_N at the north pole N;* then the $\phi(p)$ go into directions $\phi'(p)$ pointing away from N. As we run around the projection of E, these directions make one complete turn in the same direction, that is, in the positive sense as given by σ_N. Hence $F^2(\sigma_N)=1$. Similarly, projecting onto the tangent plane at S, we find $F^2(\sigma_S)=1$. Hence

$$Z^2 \cdot F^2 = F^2(\sigma_N) + F^2(\sigma_S) = 2 \neq 0. \tag{11}$$

It follows that a continuous field of directions on S^2 is impossible.

Similarly. for the n-sphere S^n, we find $Z^n \cdot F^n = 2$ if n is even and $=0$ if n is odd.† For any closed M^n (orientable or not), $Z^n \cdot F^n$ is its Euler-Poincaré characteristic χ ‡ By definition of the n-class, this equals the "index sum of singularities" of a vector field in M^n. As $2h^n=0$ for n odd, it follows that $2\chi=0$, $\chi=0$, for n odd, as is well known.

If $M^n=M^3$ is closed and orientable, then h^2 vanishes also, and it follows easily that the tangent sphere-space to any closed orientable 3-manifold is simple.§

(b) Consider first the normal space to a surface (say a Möbius strip) in E^3. Let K be a subdivision of the surface. To find F^1, we choose at each vertex a one of the two normals aa'. Suppose that for each 1-cell ab, we join the end points a' and b' of the respective normals by an arc which lies close to ab (see Fig. 1). Then clearly $F^1(ab)$ is the number of intersections (mod 2) of $a'b'$ with the surface. Thus F^1 may be found as follows. Deform the 1-dimensional part K^1 of K slightly so that the new positions a' of the vertices a are off the surface; then $F^1(ab)$ is the "Kronecker index" (mod 2) of the deformed $a'b'$ with the surface.

More generally, consider an M^n in an M^m with $m \leqq 2n$; if $m>n+1$, we assume that both are orientable, so that the nor-

* Taking S_0^1 in P_N, we may define ξ_{σ_N} as follows. Any direction d at a point p of σ_N projects into a direction d' in P_N, and determines thus a point d^* of S_0^1; we set $\xi_{\sigma_N}(p, d^*)=d$.

† It is not known whether or not the tangent space of S^n is simple for all odd n.

‡ See (for the orientable case) Stiefel [11], §5, or Alexandroff-Hopf, *Topologie* I, p. 549. The associated complex K^* of M^n has a fundamental n-cycle even if M^n is non-orientable.

§ See Stiefel [11], §5.

mal space is orientable. To find a characteristic cocycle F^{m-n}, take a subdivision K of M^n, and deform the $(m-n)$-dimensional part into a new position K'^{m-n} in M^m so that K'^{m-n-1} does not touch M^n. Then $F^{m-n}(\sigma^{m-n})$ is the Kronecker index $(\sigma^{m-n}\cdot M^n)$ of σ^{m-n} with M^n in M^m; the coefficients are integers unless $m=n+1$. For any chain A^{m-n}, $A^{m-n}\cdot F^{m-n}$ is the Kronecker index $(A^{m-n}\cdot M^n)$, as this is true if A^{m-n} is an $(m-n)$-cell.

As an application, suppose M^n is a closed orientable manifold in E^m. By joining each point of M^n to a fixed point in E^m, we construct a "singular chain" Y^{n+1} bounded by M^n (or more properly, by the fundamental n-cycle Z^n of M^n). Set $s=m-n$, and

$$G^{s-1} = (-1)^s\sum_j (\sigma_j^{s-1}\cdot Y^{n+1})\sigma_j^{s-1}. \tag{12}$$

Then, by a simple property of Kronecker indices (see for instance Lefschetz, *Topology*, page 169, (20)),

$$\begin{aligned}\delta G^{s-1}\cdot\sigma_i^s &= G^{s-1}\cdot\partial\sigma_i^s = (-1)^s\sum_j (\sigma_j^{s-1}\cdot Y^{n+1})^s\partial_i^j\\ &= (-1)^s(\partial\sigma_i^s\cdot Y^{n+1}) = (\sigma_i^s\cdot M^n) = F^s\cdot\sigma_i^s,\end{aligned}$$

and it follows that $F^{m-n}=\delta G^{m-n-1}$, and $h^{m-n}=0$.

Consider two particular cases. In $m=2n$, the normal space is 1-simple, and it follows that there is a continuous field of normals to any closed orientable M^n in E^{2n}.* If $m=n+2$, then the 2-class vanishes; as the $S(p)$ are 1-dimensional and the normal space is orientable, it follows easily that $S(M^n)$ is simple.†

We remark that any h^r may be studied for M^n in E^m, $(m>n+r)$, by projecting into E^{n+r}.‡ Further using methods in [**14**], we may construct a closed orientable M^4 in E^9 containing an S^2 with $S^2\cdot F^2\equiv 1 \pmod 2$; thus lower dimensional invariants may be non-vanishing.

(c) For an S^2 in E^3 or in E^4, the tangent space has the invariant 2, while the normal space is simple. With this in mind, we shall look only for relations between the invariants of the two spaces after reducing mod 2. A complete relation for orien-

* The theorem is probably not true in the generality stated in [**14**], footnote 25.

† See Seifert [**10**], Satz 1, and Whitney [**15**], §8, (*d*).

‡ See Whitney [**15**], §9.

tability was given at the end of §4. For an orientable M^n in an orientable M^m, it may be shown that the same statement holds for the characteristic 2-classes, after reducing mod 2. This is probably not true for the non-orientable case. (Consider a projective plane in E^4.) For still higher dimensional invariants, no complete results are known. But if the normal or tangent space is simple, the duality theorem holds for any dimension.* It follows immediately, by results in (a) and (b), that a closed orientable manifold M^n with odd characteristic χ cannot be imbedded in E^{n+2}.† This is the case, for instance, for the complex projective plane, for which $\chi=3$.

(d) Consider two sphere-spaces $S^\mu(K)$ and $S^\nu(K)$ with the same base space K. There is a unique sphere-space $S^{\mu+\nu+1}(K)$, their *product*, in which the first two spaces may be imbedded so that for each p in K, $S^\mu(p)$ and $S^\nu(p)$ are orthogonal great spheres in $S^{\mu+\nu+1}(p)$. As an example, for M^n in M^m, the product of the tangent and normal spaces gives the part $S^{m-1}(M^n)$ of the tangent space of M^m over M^n. A sphere-space which may be expressed as a product is *reducible*. A simple sphere-space is *completely reducible*, that is, it is a product of $(\nu+1)$ 0-sphere-spaces; any even number of these may be taken as non-orientable, the rest being simple. A k-simple sphere-space is the product of k simple 0-sphere-spaces and another space. See also §9.

The normal space to a closed orientable M^n in E^{n+1} is of course simple. But, one may ask, in what manner does this space lie in the tangent space of E^{n+1}? The outward normal at each point p of M^n determines a point $\psi(p)$ of the unit sphere S_0^n in E^{n+1}; thus M^n is mapped by ψ into S_0^n. The degree of ψ is the *generalized curvatura integra* of M^n in E^{n+1}.‡ If n is even, it is independent of the imbedding of M^n in E^{n+1}, and equals one half the characteristic of M^n.

For another problem on the imbedding of one sphere-space in another, consider a manifold M^n with a boundary B in a manifold M^m. Suppose a vector distribution is given over B; can it be extended over M^n?§ We may also consider sets of vec-

* See Stiefel [11], §6, No. 2.

† See Seifert [10], Satz 2.

‡ See H. Hopf [2], for various results in the direction noted.

§ See Morse [5].

tors, and so on. Of course, in defining the invariants, we considered exactly this problem, with M^n replaced by a single cell σ. The general problem of discussing how one sphere-space lies in another is almost untouched.

III. General properties of sphere-spaces

7. *On the Classification of Sphere-Spaces.* We shall now study in how far the invariants characterize a sphere-space with a given base space. But first we must clear up a matter of definition. We have allowed changes of coordinate systems in a sphere-space, considering the space to be unchanged thereby. We shall say that two sphere-spaces $S(K)$ and $S'(K)$ with the same base space K are *equivalent* if there is a continuous function $f(p)$ in K, such that, for each p, $f(p)$ is an orthogonal map of $S(p)$ into $S'(p)$. In other words, there is a homeomorphism between $\mathfrak{S}(K)$ and $\mathfrak{S}'(K)$ in which $S(p)$ and $S'(p)$ correspond for each p, the homeomorphism being orthogonal there. Then if we identify corresponding points of corresponding spheres, one sphere-space is obtained from the other merely by changing coordinate systems. If $S(K)$ and $S'(K)$ are both oriented, and there is an $f(p)$ which preserves orientations, we call them *positively equivalent.*

If K is an open or closed cell, then its cohomology groups vanish, so there can be no invariants as above described. But also any $S(K)$ is necessarily simple. It will turn out that the invariants characterize sphere-spaces also whenever $\nu=0$ or 1, and whenever dim $(K)\leqq 3$.

If $\nu=0$, there is just one type of $S(K)$ for each 1-cohomology class of K with coefficients mod 2. Suppose now that $\nu=1$ and $S(K)$ is oriented; if not, we use the methods in the footnote to §5. To show that $S(K)$ is characterized by its 2-class, suppose that $S'(K)$ has the same 2-class; we must prove $S(K)$ and $S'(K)$ to be positively equivalent, that is, we must find the map f of $S(K)$ into $S'(K)$, preserving the orientation of the $S(p)$. We first choose projections $\phi_1(p)$ and $\phi_1'(p)$ of K^1 into $\mathfrak{S}(K^1)$ and $\mathfrak{S}'(K^1)$ so as to determine the same $F^2=F'^2$. As $S(K)$ is oriented, we may choose a projection $\phi_2(p)$ of K^1 into $\mathfrak{S}(K^1)$ orthogonal to the first, so that, for each p, the ordered pair $\phi_1(p)$, $\phi_2(p)$ determines the positive orientation of $S(p)$. Do the same for $S'(p)$. These projections determine coordinate systems $\zeta(p)$ and $\zeta'(p)$

throughout K^1 with $\zeta(p, q_i) = \phi_i(p)$, $(i = 1, 2)$, and similarly for ζ'; set

$$f(p) = \zeta'(p)\zeta^{-1}(p) \quad \text{in} \quad K^1. \tag{13}$$

Now take any 2-cell σ. For each p in $\partial\sigma$, set

$$\psi(p) = \xi_\sigma'^{-1}(p)f(p)\xi_\sigma(p); \tag{14}$$

this maps S_0^1 into $S(p)$, then into $S'(p)$, and then back into S_0^1, and is a rotation. Thus $\psi(p)$ is a point of the group G^2 of rotations of S_0^1 into itself; topologically, G^2 is a circle. It is easily seen that $F'^2(\sigma) = F^2(\sigma)$ implies that the degree of ψ is 0.* Hence we may extend ψ throughout σ so that it is continuous, and define f there by inverting (14):

$$f(p) = \xi_\sigma'(p)\psi(p)\xi_\sigma^{-1}(p). \tag{15}$$

Thus f is extended throughout K^2. The boundary of a 3-cell is a 2-sphere, and any map of a 2-sphere into a circle G^2 may be shrunk to a point;† hence the same process extends f throughout K^3, and so on.

Next take any ν, and suppose dim $(K) \leqq 3$. The above proof applies again, with G^2 replaced by $G^{\nu+1}$. That f may be extended through each σ^3 follows from the fact that any map of a 2-sphere into any $G^{\nu+1}$ may be shrunk to a point.‡ What has happened to the 3-class, which should differentiate further between sphere-spaces? The above proof shows that it can play no role. In fact, given the 2-class, the sphere-space (over K^3) is completely determined, and hence so is the 3-class.§

In the next simplest case, $\nu = 2$ and dim $(K) = 4$, the invariants are insufficient. To show this, note that for $K = S^4$, the cohomol-

* Define C and D in $\mathfrak{S}(\partial\sigma)$, using ξ_σ and ζ, as in §5, and define C' and D' similarly. Under f, D goes into D', while C goes into C^* say. As $F(\sigma) = F'^1(\sigma)$, C cuts D, and hence C^* cuts D', the same algebraic number of times that C' cuts D'. Hence C^* cuts C' zero times algebraically, $\psi(p, q_1)$ cuts q_1 in S_0^1 zero times algebraically, and the degree of ψ is 0.

† See for instance Alexandroff-Hopf, *Topologie* I, p. 516, Satz II.

‡ This does not seem to have been published yet. It has been proved independently by E. Cartan, Ch. Ehresmann, and myself.

§ The exact relation between h^2 and h^3 is as follows. As F^2 is a cocycle (mod 2), if we choose G^2 with integer coefficients so that reducing mod 2 gives F^2, then $\partial G^2 = 2F^3$ for some F^3; then F^3 is a characteristic 3-cocycle.

ogy groups of dimensions 1, 2, and 3 all vanish, so that there can be no invariants of the above sort. Now let σ_1 and σ_2 be the 4-cells into which S^4 is divided by a great 3-sphere τ. To define abstractly an $S(S^4)$, we suppose that coordinate systems ξ_{σ_1} and ξ_{σ_2} are given, and we shall define the relation between them, that is, the orthogonal maps $\phi(p)=\xi_{\sigma_2}^{-1}(p)\xi_{\sigma_1}(p)$ of S_0^2 into itself (p in τ). We must thus choose a map ϕ of the 3-sphere τ into the group G^3. As the latter is homeomorphic with projective 3-space, whose covering space is the 3-sphere, these maps may be chosen in an infinity of distinct ways, thus defining an infinity of non-equivalent sphere-spaces. To define a 4-dimensional invariant distinguishing between them, we must use, for instance, the "homotopy groups" of maps of a 3-sphere into a 2-sphere as coefficient group in the 4-chains considered.

To classify sphere-spaces with $\nu=2$ and dim $(K)=4$, more advanced methods must be used.*

8. *An Imbedding Theorem.* We shall show that all possible sphere-spaces can be generated from certain simple ones. If $S(K)$ is a sphere-space and K' is any complex (or point set), then each continuous map ϕ of K' into K generates a sphere-space $S'(K')$ as follows. For each p' in K', let $S'(p')$ be the sphere $S(\phi(p'))$.

The spaces $S[\nu, \mu]$ of great ν-spheres in the μ-sphere S_0^μ† *form a set of universal sphere-spaces*; any $S^\nu(K)$ may be generated by mapping K suitably into the base space of $S[\nu, \mu]$, with $\mu=\nu+\dim (K)$.‡ We shall illustrate the proof for $\nu=1$. For each vertex a of K, let $\phi(a)$ be an arbitrary point of the base space of $S[1, \mu]$. Let $f(a)$ be an orthogonal map of $S(a)$ into the corre-

* Such methods have been found, but the classification has not been carried through. Similar methods were used in classifying the maps of a 3-complex into a 2-sphere; see Whitney, this Bulletin, vol. 42 (1936), p. 338.

† Note that the space $S[1, 2]$ of (non-oriented) great circles on S_0^2 is equivalent to the tangent space of P^2; the points on each circle give the directions from one of the corresponding poles. The base spaces of the $S[\nu, \mu]$ have been studied recently by C. Ehresmann, Journal de Mathématiques Pures et Appliquées, vol. 16 (1937), pp. 69–100. If we consider only oriented sphere-spaces, we may use the space of oriented great ν-spheres in S_0.

‡ If K is a compact metric space of finite dimension, we may easily prove the theorem by imbedding K in some E^2, defining a sphere-space with an open set in E^2 containing the image of K for base space, and applying the theorem to this space.

sponding sphere $S(\phi(a))$. We next extend ϕ and f over any 1-cell ab of K. If we define $f(p, q_1)$ and $f(p, q_2)$ for p in ab so that they are orthogonal points of S_0^μ for each p, and attach to the given values at a and b, then there is a unique orthogonal map $f(p)$ of $S(p)$ into a great circle of S_0^μ which takes on the values already given; this defines $\phi(p)$ also.

We may let $f(p, q_1)$ run along any curve on S_0^μ from $f(a, q_1)$ to $f(b, q_1)$ as p runs along ab. Now for each p in ab, there is a great $(\mu-1)$-sphere $S^{\mu-1}(p)$ of points on S_0^μ orthogonal to $f(p, q_1)$; $f(p, q_2)$ must lie on this sphere. These spheres form a sphere-space $S^{\mu-1}(ab)$, which is simple, as ab is a 1-cell (see §7). With the help of a coordinate system $\zeta(p, q)$ in it, it is easy to join $f(a, q_2)$ to $f(p, q_2)$ by the required arc, as $\mu-1\geqq 1$. The extension of f through cells of higher dimension is carried out in the same manner.

9. *On Simple Sphere-Spaces.* If $S(K)$ is simple, it can be expressed as a product $K\times S_0^\nu$, that is, a coordinate system $\zeta(p, q)$ may be defined for all p in K. In how many ways may we do this, preserving the orientation of the $S(p)$? To answer this, let ζ_0 be a fixed coordinate system. Then any other one, ζ, determines a rotation $\phi(p)=\zeta^{-1}(p)\zeta_0(p)$ of S_0^ν; there is thus defined a continuous map ϕ of K into $G^{\nu+1}$. Conversely, any ϕ determines a new coordinate system ζ from the given one ζ_0. Thus *the expressions of $S(K)$ as a product, preserving orientation, correspond to the maps of K into the group $G^{\nu+1}$ of rotations in $(\nu+1)$-space.*

For an example, take $\nu=1$, K a circle; $\mathfrak{S}(K)$ is a torus. G^2 is a circle, and there are an infinity of non-homotopic maps and corresponding coordinate systems. For $\nu=1$ and any K, the essentially distinct expressions of $S(K)$ as a product correspond to the elements of the 1-dimensional cohomology group of K with integer coefficients.

IV. r-FUNCTIONS IN A DIFFERENTIABLE MANIFOLD

10. *An Imbedding Problem; Vector Functions.* We begin by considering the problem of imbedding a differentiable manifold M^n in euclidean space. Suppose f is a (continuously) differentiable map of M^n into E^m. Then $f(p)$ has m coordinates $f_1(p), \cdots, f_m(p)$, and if we use a coordinate system $(x_1, \cdots, x_n)$

in a region of M^n, each $f_j(p)=f_j(x_1, \cdots, x_n)$ is differentiable. The ith contravariant coordinate vector at p is mapped into a contravariant vector in E^m with components $\partial f_1/\partial x_i, \cdots, \partial f_m/\partial x_i$. If these n vectors, $(i=1, \cdots, n)$, are independent, then clearly a neighborhood of p goes into E^m in a (1-1) way. If this holds for each p in M^n, we call the map *regular*. It is regular at p if the matrix

$$\left\| \begin{array}{ccc} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_1} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \dfrac{\partial f_1}{\partial x_n} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{array} \right\|, \qquad \begin{array}{l} i\text{th row} = \text{map of } i\text{th coordinate vector}, \\ \\ j\text{th column} = \text{gradient } \nabla f_j \text{ of } f_j, \end{array}$$

is of rank n. The jth gradient ∇f_j, a covariant vector in M^n, may be considered as the "dual map" into M^n of the jth covariant coordinate vector of E^m. *A regular map is always possible if* $m \geqq 2n$ (see [14], Theorem 3). To study the question for $m<2n$, perhaps the best method is to look for m covariant vector functions in M^n such that (a) they are gradients, and (b) at each point of M^n, n of them are independent.*

We are thus led to the problem, when is a given (covariant) vector function v a gradient? As is well known, v is a gradient if and only if its integral $\int_C v=\int_C \sum v_i dx_i$ around every closed curve C in M^n vanishes. Now let us ask, when is a vector function such that integrating around any curve gives the same result as integrating around any sufficiently nearby curve? The answer is, if and only if the integral around any sufficiently small curve is 0; or, the function is *exact*, that is, all numbers $w_{ij}=\partial v_i/\partial x_j-\partial v_j/\partial x_i$ in any coordinate system in M^n vanish. In any simply connected region, a vector function is a gradient if and only if it is exact; but this does not hold in general over the whole manifold. For the relation of these functions to chains in a subdivision of the manifold, and generalization to higher dimensions, see the next section.

For a thorough study of gradients ∇f and their singularities

* If we use [14], Theorem 4, we need not distinguish between contravariant and covariant vectors.

(critical points of f), we refer to the work of Morse and others.* Almost no study has been made yet of sets of gradients.†

11. *r-Functions in a Manifold.*‡ By an *r-function* in a differentiable manifold M^n we shall mean a differentiable§ alternating covariant tensor α of order r. We may find the integral $\int_\sigma \alpha$ of α over an r-cell σ; then we define the integral over an r-chain by

$$(16) \qquad \int_{\Sigma a_i \sigma_i^r} \alpha = \sum a_i \int_{\sigma_i^r} \alpha.$$

Corresponding to any r-function α there is a *derived* $(r+1)$-function α; α is *exact* if $\alpha' = 0$. A 0-function is a scalar function; as its derived is its gradient, it is exact if and only if it is a constant (at least if M^n is connected). For a vector function, that is, a 1-function, with components v_i, the derived has the components w_{ij} as given above. The function α is *derived* if it is the derived of an $(r-1)$-function β. As $(\alpha')' = 0$, any derived function is exact. By Stokes's theorem, for any r-function α and any $(r+1)$-chain A,

$$(17) \qquad \int_{\partial A} \alpha = \int_A \alpha';$$

for this holds for any σ^r. This corresponds to (4).

Take any subdivision K of M^n. To each r-function α corresponds an r-chain, with *real numbers* for coefficients, defined by

$$(18) \qquad \phi(\alpha; \phi_i)^r = \int_{\sigma_i^r} \alpha, \qquad \phi(\alpha) = \sum_i \phi(\alpha; \sigma_i^r)\sigma_i^r;$$

then $\int_A \alpha = A \cdot \phi(\alpha)$. By (7) and (4), for any $\sigma = \sigma^{r+1}$,

$$\phi(\alpha') \cdot \sigma = \int_\sigma \alpha' = \int_{\partial\sigma} \alpha = \phi(\alpha) \cdot \partial\sigma = \delta\phi(\alpha) \cdot \sigma,$$

and hence, as an equivalent of Stokes's theorem,

$$(19) \qquad \phi(\alpha') = \delta\phi(\alpha).$$

* See for instance M. Morse, *The Calculus of Variations*, Ch. VI.

† See A. E. Pitcher, this Bulletin, vol. 43 (1937), p. 167.

‡ Compare de Rham [6], [7], and [8], and, for non-continuous functions with general coefficient groups, J. W. Alexander, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 511–512.

§ The theory holds if we use tensors with continuous partial derivatives of order k, that is, of class C^k, the derived tensors being of class C^{k-1}.

From this formula it is apparent that *if α is exact, then $\phi(\alpha)$ is a cocycle, and if $\beta=\alpha'$ is derived, then $\phi(\beta)$ is a coboundary.*

We shall need an existence lemma: *If α is an exact r-function, and B is an $(r-1)$-chain such that $\delta B=\phi(\alpha)$, then there is an $(r-1)$-function β such that $\phi(\beta)=B$ and $\beta'=\alpha$.* We define β over K cell by cell, using an extended form of a lemma of de Rham* and a theorem on the extension of differentiable functions.† From this we deduce two facts: (a) If α is exact and $\phi(\alpha)$ is a coboundary, then α is derived. For B exists. (b) For any cocycle B there exists an exact function β with $\phi(\beta)=B$. For we may use $\alpha\equiv 0$.

We now prove the fundamental theorem: *If we identify any two exact r-functions whose difference is derived, we obtain the rth cohomology group of K with real coefficients.* We map each class of exact functions α into the cohomology class of $\phi(\alpha)$; this is easily seen to be a well defined homeomorphism. That it is actually an isomorphism follows at once from (a) and (b) above.

Each cocycle A may be considered as a linear function L defined on cycles, with $L(\partial B)=0$, if we set $L(B)=A\cdot B$. The group described is isomorphic with the set of all such functions. Hence, if $C_1, \cdots, C_p$ form a maximal set of independent cycles (p is the rth Betti number of the manifold), the integrals $\int_{C_i}^{\alpha}$ may be chosen arbitrarily, α being determined up to derived functions. These integrals are called the *periods* of α.

12. *On Products of Functions.*‡ Given an r-function α and an s-function β, we define as usual their (alternating) product, giving an $(r+s)$-function $\alpha\beta$. We recall that

$$(\alpha\beta)' = \alpha'\beta + (-1)^r\alpha\beta'. \tag{20}$$

Also, chains of dimensions r and s may be multiplied, giving a chain of dimension $r+s$; if certain simple conditions are satisfied, including

$$\delta(A \cup B) = \delta A \cup B + (-1)^r A \cup \delta B, \tag{21}$$

* See de Rham [6], §21, Lemma II; we do not need his Lemma III in our method of proof.

† Whitney, Transactions of this Society, vol. 36 (1934), pp. 63–89, Lemma 2.

‡ Compare de Rham [6], §27, and [7]. The proof below clearly holds for both orientable and non-orientable manifolds.

then there is a uniquely defined corresponding product in the cohomology groups.* We wish to show that these products correspond.

First, it is possible to construct† an r-function $\omega(\sigma^r)$ corresponding to each r-cell σ^r (for all r), such that $\omega(\sigma^r)$ vanishes outside the star of σ^r, and setting $\omega(\sum a_i\sigma_i^r)=\sum a_i\omega(\sigma_i^r)$, we have

$$(22) \qquad \omega'(A) = \omega(\delta A), \qquad \phi(\omega(A)) = A, \qquad (\text{all } A).$$

Also, if I is the sum of the vertices of K, then $\omega(I)\equiv 1$.

Now to each r-chain A and s-chain B we construct a corresponding $(r+s)$-chain

$$(23) \qquad A \cup B = \phi[\omega(A)\omega(B)].$$

Applying (19) and (20), we derive (21). Also $I \cup B=B$. Noting also a certain local condition, we see that it follows that the product thus defined has the required properties. We now prove easily in turn: (a) If α is exact, then so is $\omega(\phi(\alpha))$, and $\phi[\alpha-\omega(\phi(\alpha))]=0$, so that $\alpha-\omega(\phi(\alpha))$ is derived. (Use (a) of §11.) (b) If β is exact also, then $\alpha\beta-\omega(\phi(\alpha))\beta$, and also $\alpha\beta-\omega(\phi(\alpha))\omega(\phi(\beta))$, are derived. (Use (20).) Finally,

$$(24) \qquad \phi(\alpha\beta) \backsim \phi[\omega(\phi(\alpha))\omega(\phi(\beta))] = \phi(\alpha) \cup \phi(\beta).$$

Hence *the products of functions and the products of cocycles determine the same products in the cohomology groups.*

BIBLIOGRAPHY

We give here only those papers most closely connected with the present paper.

1. Hadamard, J., *Sur quelques applications de l'indice de Kronecker*, Appendix to J. Tannery, *Théorie des Fonctions*, Paris, 1910.

2. Hopf, H., *Über die Curvatura Integra geschlossener Hyperflächen*, Mathematische Annalen, vol. 95 (1925), pp. 340–367.

3. Hopf, H., *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Mathematische Annalen, vol. 96 (1927), pp. 225–250.

4. Hotelling, H., *Three dimensional manifolds of states of motions*, Transactions of this Society, vol. 27 (1925), pp. 329–344.

* See Whitney, PC, §11; also Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 285–291, Theorem 4.

† See de Rham [6], §24.

5. Morse, M., *Singular points of vector fields under general boundary conditions*, American Journal of Mathematics, vol. 51 (1929), pp. 165–178.

6. de Rham, G., *Sur l'analysis situs des variétés à n dimensions*, Journal de Mathématiques Pures et Appliquées, (10), vol. 9 (1931), pp. 115–200.

7. de Rham, G., *Sur la théorie des intersections et les intégrales multiples*, Commentarii Mathematici Helvetici, vol. 4 (1932), pp. 151–157.

8. de Rham, G., *Relations entre la topologie et la théorie des intégrales multiples*, l'Enseignement Mathématique, vol. 35 (1935), pp. 213–228.

9. Seifert, H., *Topologie dreidimensionaler gefaserter Räume*, Acta Mathematica, vol. 60 (1932), pp. 147–238.

10. Seifert, H., *Algebraische Approximation von Mannigfaltigkeiten*, Mathematische Zeitschrift, vol. 41 (1936), pp. 1–17.

11. Stiefel, E., *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Commentarii Mathematici Helvetici, vol. 8 (1936), pp. 3–51.

12. Threlfall, W., *Räume aus Linienelementen*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 42 (1932), pp. 88–110.

13. Wazewski, W., *Sur les matrices dont les éléments sont des fonctions continues*, Compositio Mathematica, vol. 2 (1935), pp. 63–68.

14. Whitney, H., *Differentiable manifolds*, Annals of Mathematics, vol. 37 (1936), pp. 645–680.

15. Whitney, H., *Sphere-spaces*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 64–68.

The work of Hopf, Rinow, Myers, and Cohn-Vossen on the relations between the topology and differential geometry of analytic Riemannian manifolds should also be noted.

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