

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

372. Professor R. P. Agnew: *Linear functionals satisfying prescribed conditions.*

Let P be the class of p -functions $p = p(x)$ and F the class of linear functionals $f = f(x)$, defined over a linear space E . A functional $p \in P$ is said to enforce a set S of properties if each $f \in F$ with $f(x) \leq p(x)$ for all $x \in E$ has the properties S . It is shown that $p \in P$ exists which enforces S if and only if $f \in F$ exists having properties S , and that $p_2 \in P$ enforces all the properties which $p_1 \in P$ enforces if and only if $p_2(x) \leq p_1(x)$ for all $x \in E$. The main problem of this paper is that of characterizing analytically the class of $p \in P$ which enforces a preassigned property or set of properties. The problem is solved for the property $1^\circ: f(x) \leq p_0(x)$, $p_0 \in P$ being preassigned; for the property $2^\circ: f(y) = f(x)$ for all pairs $\{x, y\}$ belonging to a preassigned set Ψ of pairs; and for the property 3° : both 1° and 2° . The analytic criteria are discussed for the case where Ψ is the set of all pairs $\{x, g(x)\}$ with $x \in E$, and $g \in G$ where G is a group of linear transformations mapping E into itself, and for the further special case where G is solvable. Applications to generalized limits and integrals are given. (Received September 25, 1937.)

373. Dr. R. P. Boas and Professor D. V. Widder: *The iterated Stieltjes transform.*

The iterated Stieltjes transform is (1) $f(x) = \int_{0+} (x+u)^{-1} du \int_{0+}^{\infty} (u+t)^{-1} d\alpha(t)$. It is formally equivalent to the S_2 -transform (2) $f(x) = \int_{0+}^{\infty} \log(x/t)(x-t)^{-1} d\alpha(t)$; this formal equivalence is not always valid, but (2) can always be written in the form (1). The transform (1) is inverted by the use of a differential operator $H_{k,i}[f(x)] = L_{k,i} L_{k,i}[f(x)]$, where $L_{k,i}[f(x)]$ is the inversion operator for the Stieltjes transform, introduced by D. V. Widder (abstract 43-1-87). If $\alpha(t)$ is the integral of a function $\phi(t)$, $H_{k,i}[f(x)]$ approaches $\phi(t)$ for almost all positive t , ($k \rightarrow \infty$). If $\alpha(t)$ is a normalized function, of bounded variation on every interval (ϵ, R) , ($0 < \epsilon < R < \infty$), the integral of $H_{k,i}[f(x)]$ on $(0, t)$ approaches $\alpha(t) - \alpha(0+)$ for every positive t . Necessary and sufficient conditions for a given $f(x)$ to have the representations (1) and (2) with $\alpha(t)$ belonging to various functional classes are obtained in terms of $H_{k,i}[f(x)]$. For a preliminary report on the S_2 -transform, see D. V. Widder, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 242-244. (Received September 18, 1937.)

374. Professor Richard Brauer and Dr. C. J. Nesbitt: *On modular representations of groups of finite order.*

Any irreducible representation of a group G of finite order g is equivalent to a representation A whose coefficients are integers of an appropriate algebraic number field K . If p is a rational prime, we obtain a modular representation $A^{(p)}$ of G on considering A modulo a prime ideal of K which divides p . It is well known that $A^{(p)}$ is irreducible if p does not divide the group order g . This result will be extended in the following way to the singular case where p divides g . Let p^a be the highest power of p dividing g . If p^a divides the degree of A , then $A^{(p)}$ is an irreducible modular representation. Two non-equivalent ordinary representations A_1 and A_2 which satisfy the assumption yield non-equivalent modular representations. No such $A^{(p)}$ appears as irreducible constituent of an indecomposable modular representation U (except for $U = A^{(p)}$). (Received September 25, 1937.)

375. Professor R. H. Cameron: *The distribution of values of an analytic almost periodic function in equally spaced circles.*

This paper deals with functions analytic in a strip and almost periodic in a substrip. A system of equally spaced circles is drawn with their centers on the edge of the almost periodic strip, and it is shown that the function takes on values arbitrarily close to a given value A in a relatively dense set of these circles. If the distance between the circles belongs to the module of the function, an extra restriction on the set of circles is necessary. (Received October 4, 1937.)

376. Dr. J. H. Curtiss: *Uniform convergence and summability of the Jacobi series on an unrestricted lemniscate.*

Integrals of the Dirichlet and Fejér type are set up for the real and imaginary parts of the Jacobi interpolation series, and are studied in some detail. It is found, for example, that if the boundary values $F(z)$ of the sum function are continuous on the lemniscate of convergence, then the Jacobi series is summable (C, α) , $\alpha > 0$, to $F(z)$ uniformly on the lemniscate of convergence, regardless of whether or not the lemniscate has multiple points. Uniform convergence occurs if the boundary values are continuous and of bounded variation. The degree of uniform convergence corresponding to a Lipschitz condition is also investigated by real variable methods, and examples are given to show that the results are the best possible. (Received October 4, 1937.)

377. Mr. J. J. DeCicco: *Fields whose geodesic series have circles for their point- or line-unions.*

This paper is a continuation of the two papers, *The group of turns and slides and the geometry of turbines*, by Kasner, American Journal of Mathematics, vol. 33 (1911), *The geometry of turbines, flat fields, and differential equations*, by Kasner and the author, American Journal of Mathematics, vol. 59 (1937), and abstract 43-5-260, *The differential geometry of series of lineal elements*, by the author. A series of a field is called a *geodesic series* when its curvature at any

element is less than the curvature of the other field series which pass through that element and have the same tangent turbine. The main results are: 1. For ∞^2 circles to be the point curves of the geodesic series of a field, it is necessary and sufficient that they all pass through a fixed point. Moreover the field is a non-linear flat field. 2. For ∞^2 circles to be the line curves of the geodesic series of a field, it is necessary and sufficient that they be tangent to a fixed (linear or non-linear) circle. (Received September 27, 1937.)

378. Professor Philip Franklin: *Note on the four color problem.*

Recently, the author announced the theorem that every regular map, irreducible with respect to four colors, contains at least 32 regions. In view of certain new reducible configurations and results due to C. E. Winn, the proof of this theorem is considerably simplified. (Received September 24, 1937.)

379. Dr. V. V. Latshaw: *On fourth order self-adjoint difference systems.*

This note includes three theorems: one on adjoint functions, one on self-adjoint boundary conditions, and one on characteristic values. (Received September 30, 1937.)

380. Dr. A. N. Lowan: *Note on the operational determination of Green's functions in the theory of heat conduction.*

This note is a sequel to a paper appearing in the July issue of the Philosophical Magazine and bearing a similar title. It deals with the determination of the Green's function for a cylinder and for the space bounded internally by a cylinder, the boundary condition in both cases being in the form $(\partial/\partial r + h)u = 0$ from which the corresponding solutions for the cases where the boundary is either impervious to heat or kept at 0 degrees is obtained by putting $h = 0$ and $h = \infty$, respectively. The Green's function is constructed by the superposition of the solution of the differential equation of heat conduction satisfying the condition for a line source and one vanishing at $t = 0$. The Laplace transforms of the last two solutions are first obtained, the arbitrary constants being determined so as to satisfy the boundary condition. By means of Neumann's expansion and some known identities involving integrals of Bessel's functions, the sum of the last two Laplace transforms is written in the form of an integral whose inverse Laplace transform is readily obtained. (Received September 29, 1937.)

381. Professor Deane Montgomery and Dr. Leo Zippin: *On abelian groups of auto-homeomorphisms of metric spaces.*

This note is concerned with compact abelian groups of auto-homeomorphisms of a metric space E from the point of view, particularly, of the "orbit-space." It gives a useful characterization of the dimension of compact abelian groups and it contains the following quite special theorem: the topological rotation group of the euclidean 3-space E is characterized as an abelian compact connected group of auto-homeomorphisms of E . (Received October 2, 1937.)

382. Professor Tadası Nakayama: *A remark on representations of groups.*

In this paper the Frobenius theorem on induced characters of finite groups is extended to the case of bounded representations of general groups. The extension is based on the fact that, with a suitable definition of compositions, we can consider the ring of almost periodic functions on a subgroup (of a not necessarily finite index) of a group as a left or right operator-ring of that of the whole group. (Received September 29, 1937.)

383. Professor Rufus Oldenburger: *Decomposition of forms.*

Sylvester, Dickson, Bronowski, and others studied the representation of forms by sums of powers of linear forms. In the present paper it is proved that, for a field with p or more elements, every form F of degree p with symmetric matrix can be written as a sum $S = aL^p + \dots + dM^p$, where there are a finite number N of forms in the set L, \dots, M of linear forms, and a, \dots, d , are in the given field. For a field with less than p elements such a representation of F is not, in general, possible. In another paper, not yet published, it was proved that a form F , for a field for which the symmetric matrix of F is unique, can be represented by a sum S where L, \dots, M are linearly independent if and only if a determinant rank of F and a new type of invariant called "factorization rank," of F are equal. For a given form F and value of N the representation S is not unique. The relations between representations of a given form F by sums of type S , where L, \dots, M are linearly independent, can be expressed in terms of diagonal and permutation matrices. An analogous theory holds for multilinear forms. (Received October 4, 1937.)

384. Dr. B. J. Pettis: *A note on regular Banach spaces.*

Several conditions can be listed as necessary and sufficient that a Banach space \mathfrak{X} be regular, that is, that given X in $\overline{\mathfrak{X}}$ there is an x in \mathfrak{X} such that $X(f) = f(x)$ for all f in $\overline{\mathfrak{X}}$: for example, regularity of $\overline{\mathfrak{X}}$ is equivalent to that of \mathfrak{X} ; separable \mathfrak{X} is regular if and only if \mathfrak{X} is weakly complete and either $\overline{\mathfrak{X}}$ is separable or \mathfrak{X} is weakly compact. As a corollary, in the space of bounded additive functions (of L -measurable sets) satisfying condition (N) of Lusin the completely additive (N) functions form a closed set of the first category. The space of completely continuous operations from regular \mathfrak{X} to 1^1 (absolutely convergent series) is equivalent to that of the unconditionally convergent series in $\overline{\mathfrak{X}}$; if in addition \mathfrak{X} is separable this also gives the space of linear operations from \mathfrak{X} to 1^1 . The functions of bounded variation from a linear interval to a given \mathfrak{X} are all strongly differentiable almost everywhere if and only if all are weakly differentiable almost everywhere; hence if \mathfrak{X} is separable and regular, every BV function from the line to \mathfrak{X} is strongly differentiable almost everywhere and its derivative is Bochner-integrable. (Received September 7, 1937.)

385. Mr. L. B. Robinson: *On a functional equation which admits a lacunary function as solution.*

The author has studied the equation $u'(x) = \lambda/(a-x^2)u(x^2)$ when $a=1$. When $a \neq 1$, the solution converges within and without the unit circle. It also

admits the circumference of the unit circle as a natural boundary. Moreover each point of the circumference is a point of accumulation in whose neighborhood can be found an infinity of singular points. (Received August 26, 1937.)

386. Dr. J. F. Randolph: *The Vitali covering theorem for Carathéodory linear measure.*

Results on the same problem were presented to the Society under a similar title in January, 1936 (abstract 42-1-45). In this paper, the same results are obtained under weaker conditions, and necessity as well as sufficiency is discussed. (Received October 5, 1937.)

387. Dr. A. R. Schweitzer: *Remarks on the foundations of geometry.*

The author's remarks concern first the existence of additions to his sets of generating relations for descriptive and metrical geometry, such as angular analogs of the author's relations S , B , I , expressing respectively angular "incidence" and "non-incidence" and "point in the interior of the angle α_{jn} " (see abstract 43-1-92), and the corresponding metrical relations expressing angular bisection. The remaining part of the paper is devoted to the n -dimensional generalization of the archimedean postulate in terms of circles, spheres, and so on, and the corresponding statement of (so-called pure) continuity. A critical analysis is made of the postulates of non-archimedean continuity of Veronese, Hilbert, and Vahlen. In particular, Hilbert's "axiom of completeness" is replaced by an analogous but possibly more concise postulate which emphasizes the non-adjoinability of existential axioms rather than the non-addition of geometric elements. From the point of view of this replacement, Hilbert's concept of completeness has interesting anticipatory analogies in the literature of antiquity. (Received October 18, 1937.)

388. Professor Tibor Radó: *A theorem on generalized Jacobians.*

Let the continuous transformation T be defined by the equations $x = x(u, v)$, $y = y(u, v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$. Suppose T is absolutely continuous (as defined by Banach, *Fundamenta Mathematicae*, vol. 7 (1925), pp. 225-236), and denote by $J(u, v)$ the generalized Jacobian introduced by Schauder (the product $i(u, v) D(u, v)$ on p. 71 of his paper in *Fundamenta Mathematicae*, vol. 12 (1928), pp. 47-74). Suppose finally that the partial derivatives x_u , x_v , y_u , y_v exist almost everywhere. The purpose of this paper is to prove that one has then $J(u, v) = x_u y_v - x_v y_u$ almost everywhere. (Received October 7, 1937.)

389. Professor Akitsugu Kawaguchi: *Geometry in an n -dimensional space with the arc length $s = \int (A_i x'^i + B)^{1/2} dt$.*

In this paper a theory is developed in a special Kawaguchi space in which the arc length is given by $s = \int (A_i x'^i + B)^{1/2} dt$. At first the intrinsic connections C_1 and C_2 , which are invariant under any parameter transformation $\bar{t} = \bar{t}(t)$, are considered and, by use of these connections, the covariant differentiations,

curvature, and torsion tensors are then introduced. The geodesic and minimal curves are also discussed and their geometrical interpretations are added. Finally many other important and fundamental formulas, such as generalized Bianchi's identities, are also completely calculated. (Received October 6, 1937.)

390. Mr. P. R. Halmos: *Note on almost-universal forms.*

Ramanujan and Dickson proved that there are 54 universal forms $ax^2+by^2+cz^2+dt^2$, with positive integral coefficients a, b, c, d . It is the purpose of this note (which is to appear shortly in this Bulletin) to investigate *almost-universal forms*, that is, to exhibit sets of positive integral coefficients a, b, c, d such that $ax^2+by^2+cz^2+dt^2$ represents every positive integer with exactly one exception. A set of 88 forms is exhibited which has to contain all almost-universal forms. By a method involving the regularity of ternary forms it is shown that 86 of these forms are almost-universal. The method breaks down in the case of the forms $x^2+2y^2+7z^2+11t^2$ and $x^2+2y^2+7z^2+13t^2$ because of the essential irregularity of all the ternary forms involved. (Received October 14, 1937.)

391. Professor C. H. Forsyth: *Extension formulas for interpolation with leading differences.*

Interpolation by leading differences is very valuable in interpolating whole series of values, not only because of the labor saved thereby, but also because of the valuable running check upon the work provided from interval to interval. However, such work calls for the separate computation of these leading differences for each interval. The author develops extension formulas for continuing the work from interval to interval with much less labor since these extension formulas are of a higher order of differences and therefore much simpler. Formulas are developed to cover both ordinary interpolation and interpolation with groups of given values. (Received October 19, 1937.)

392. Dr. E. F. Beckenbach: *A relative of the lemma of Schwarz.*

It is shown that if $p(r, \theta)$ is non-negative, and $\log p(r, \theta)$ is subharmonic, for $r < 1$, then $q(r, \theta) \equiv \int_0^r p(\rho, \theta) d\rho$ also has a subharmonic logarithm for $r < 1$. It follows that if $w=f(z)$ is analytic for $|z| < 1$, and if for every θ , $l(1, \theta) \leq 1$, where $l(r, \theta)$ is the length of the image on the w -plane of the radius from $(0, 0)$ to (r, θ) on the z -plane, then $l(r, \theta) \leq r$. A generalization to conformal maps on surfaces of negative Gaussian curvature is obtained. (Received October 12, 1937.)

393. Dr. W. C. Randels: *On the weak summability of Fourier series.*

A method of summability defined by a matrix $\{a_{mn}\}$ is said to be weakly effective if the sequence of transforms $T_m(f, x) = \sum a_{mn} f_n e^{in\alpha}$ of the Fourier series of $f(x)$ converges weakly in L_p to $f(x)$. It is shown that weak effectiveness is equivalent to the strong effectiveness defined by Hille and Tamarkin. (Received October 12, 1937.)

394. Professor R. V. Churchill: *Additional notes on the inversion of the Laplace transformation.*

In a recent paper (Mathematische Zeitschrift, vol. 42 (1937), pp. 567-579) the author gave criteria, useful in the solution of boundary value problems, for the representation of the inverse Laplace transform by a series. These are conditions for the validity of an expansion formula which is a generalized form of one of Heaviside's formulas. Conditions which are more useful in problems involving differential equations of parabolic type are established and illustrated in the present note. (Received October 12, 1937.)

395. Professor A. A. Albert: *A quadratic form problem in the calculus of variations.*

The paper gives the first proof of a conjecture of Professor G. A. Bliss which arose in connection with a sufficiency proof in the calculus of variations. Let f and g be real quadratic forms in n real variables such that f is positive for all values of the variables making $g=0$. Then there exists a real r such that $f+rg$ is positive definite. The range for r is also determined in terms of the characteristic roots of the matrix of the pencil $f+rg$. (Received October 13, 1937.)

396. Dr. H. H. Goldstine: *Weakly complete Banach spaces.*

The author extends the definition of weakly complete spaces by means of the Moore-Smith limit and investigates the problem of determining conditions for a Banach space to be weakly complete. A necessary and sufficient condition is that the space be equivalent in the sense of Banach to the adjoint of its adjoint. Moreover, when the adjoint of the original space is separable the definition of weak completeness given by the author reduces to the one ordinarily stated. (Received October 16, 1937.)

397. Professor L. M. Blumenthal: *The characterization of pseudo- $S_{2,r}$ -sets.*

In this paper is solved the problem proposed by Karl Menger of characterizing those pseudo- $S_{2,r}$ -sets containing more than five points, no pair of points being diametral (that is, with a distance equal to πr). It is proved that the distance of every pair of distinct points of such a set equals $r \cos^{-1}(\pm 1/3)$, with the negative sign holding for a number $\mu > 0$ of the distances. As a corollary of this result the following determinant theorem is proved: If the determinant $\Delta = |r_{ij}|$, $r_{ij} = r_{ji}$, $-1 < r_{ij} < 1$, ($i \neq j$), $r_{ii} = 1$, ($i, j = 1, 2, \dots, m; m > 5$), is such that (i) every third-order principal minor is non-negative, (ii) every principal minor of order four vanishes, (iii) at least one fifth-order principal minor is not zero, then (1) every third-order principal minor is positive, (2) every principal minor of order five has the value $-(4/3)^4/3$, (3) the absolute value of each element of Δ outside the principal diagonal is $1/3$, (4) upon multiplying rows and the same numbered columns of Δ by -1 , each element of Δ outside the principal diagonal takes the value $-1/3$, (5) $\Delta = -(4/3)^{m-1}(m-4)/3$. (Received October 27, 1937.)

398. Dr. G. M. Ewing: *Some implicit function theorems in the large.*

Sufficient conditions for $f(x, y)=0$ to define y as a function of x in the neighborhood of a particular point have been given in many forms. In the present paper theorems are obtained which insure the existence of the implicit function over a specified domain (x_1, x_2) of the independent variable. The method used involves a suitable adaptation of the Hilbert construction used in existence theorems in the calculus of variations. The paper also includes a generalization to the case of n implicit functions determined by n equations. (Received October 27, 1937.)

399. Dr. P. O. Bell: *On a family of curves associated with a conjugate net.*

The author introduces a family of curves which corresponds to an arbitrarily selected conjugate net on a non-ruled analytic surface. This family is studied in detail, and particularly with reference to its relation to other families. A number of interesting results are obtained, of which the following is typical: A geometric characterization is obtained for a two-parameter family of curves whose cusp-axis is the axis of C ech. Simple geometric specializations present the curves of Segre and the curves of Darboux as unique curves of this family. (Received October 27, 1937.)

400. Professor Dunham Jackson: *Note on orthogonal polynomials in three variables.*

This is a supplement to an earlier paper by the author on the same subject (abstract 43-1-31), relating particularly to the convergence of developments in series of the orthogonal polynomials, and a proof of the uniqueness of the weight function for a given orthogonal system. (Received October 28, 1937.)

401. Mr. J. H. Daoust: *On certain constants associated with Bernstein's theorem.*

An extension of Bernstein's theorem on the derivative of a polynomial or a trigonometric sum, relating to the behavior of a trigonometric sum in an interval (a, b) of length less than a period, has been stated (D. Jackson, Transactions of this Society, vol. 40 (1936), p. 227) in a form involving a constant C which depends only on the length of the interval. The present paper shows that this theorem is true with a value of C which is not dependent even on a and b , but is an absolute constant. Some other theorems of similar character are rendered more precise in a corresponding manner. (Received October 28, 1937.)

402. Mr. Fulton Koehler: *Orthogonal polynomials on a square.*

This paper is a study of formal properties of polynomials orthogonal on the perimeter of a square as a special case of polynomials in two real variables orthogonal on an algebraic curve. The polynomials of the orthogonal system with unit weight function are found to be expressible explicitly in terms of Jacobi polynomials. The application of the formal results to problems of convergence will be dealt with in a later paper. (Received October 28, 1937.)

403. Dr. R. B. Kershner: *On a van der Corput absolute constant.*

According to van der Corput, there exist absolute constants μ, ν such that if $f(x)$ is a real function for which $f''(x) \geq r > 0$ in $[a, b]$, then (1) $|\int_a^b \cos f(x) dx| \leq \mu/r^{1/2}$, (2) $|\int_a^b e^{if(x)} dx| \leq \nu/r^{1/2}$. The author has previously determined the best value μ_0 of μ satisfying (1). It is shown that the best value ν_0 of ν satisfying (2) is given by $\mu_0 = \nu_0$. A direct determination of ν_0 is given which, in view of $\nu_0 = \mu_0$, provides an alternative, and more intuitive, determination of μ_0 . (Received October 28, 1937.)

404. Dr. H. J. Hamilton: *Some theorems on subsequences.*

Let \mathfrak{A} be the class of all sequences $\{a_k\}$ (finite or infinite) of complex numbers for which $0 < \sum |a_k| < \infty$. The following theorems are proved. (A) Given any sequence $\{a_k\} \in \mathfrak{A}$, there exists a subsequence $\{a_j^*\}$ for which $|\sum a_j^*| = \sup |\sum a'_j|$ over all subsequences $\{a'_j\}$ of $\{a_k\}$. Define $\rho(\{a_k\}) = |\sum a_j^*| / \sum |a_k|$. Then (B) $\rho(\{a_k\}) > 1/\pi$, and (C) $1/\pi = \inf \rho(\{a_k\})$ over all $\{a_k\} \in \mathfrak{A}$. Incidentally, a proof is given that the conditions $\sum |c_k| < \infty$, ψ_k is real (all k), and $\sum |c_k \cos(\phi - \psi_k)| = 0$ for all real ϕ imply that $c_k = 0$ (all k). Somewhat analogous theorems are proved for sequences $\{b_k\}$ for which $\sum |b_k| = \infty$. (Received October 22, 1937.)

405. Mr. R. P. Dilworth. *Abstract residuation over lattices.*

If A and B are ideals in a commutative ring with unit element, the residual of B with respect to A is an ideal $A:B$ having the properties $A \supset (A:B)B$; if $A \supset XB$, then $A:B \supset X$. The purpose of this paper is to give a postulational treatment of a residual defined over a lattice. Since in general no multiplication exists in a lattice, the properties which are proved for the residual are those which are independent of multiplication. Equality is taken as an undefined relation with cross-cut, union, and residual as undefined connections. As an application of the postulates, the residual is determined for a Boolean algebra. (Received October 26, 1937.)

406. Mr. N. A. Hall: *A formal expansion theory for functions defined by two variable power series.*

A generalization to several variables of the Blissard umbral notation for power series enables us to give the expansion of two variable functions in very general sequences of functions, the coefficients in the expansion being given explicitly in terms of the coefficients of the power series for the original function and the coefficients of the power series for the sequence of functions. In particular we give the expansions in products of Bessel functions, $J_n(x) \cdot J_m(y)$, in products of orthogonal polynomials, in mixed products such as Bessel functions and Sonine polynomials, and in a great variety of hypergeometric functions of two variables. (Received October 26, 1937.)

407. Mr. J. W. Green: *A property of harmonic functions in three variables.*

In the plane, a function harmonic within a circle and with a normal deriva-

tive which vanishes on an open set on the circle is uniquely extensible across the open set to every point exterior to the circle. In three dimensions the analogous result does not hold, and an example is exhibited to prove this fact. (Received October 29, 1937.)

408. Mr. D. W. Hall and Dr. G. E. Schweigert: *Properties of invariant sets under pointwise periodic homeomorphisms.*

Let M be a compact separable metric space and T a pointwise periodic homeomorphism of M into itself. Let a point group of M under T be defined as the finite subset of M consisting of a point x of M and all of its images under T . The following results are obtained: (1) Every closed subset L of M which is invariant under T is either vacuous, connected, or has the property that for every possible representation of L as the sum of two mutually separated sets A and B , both A and B are invariant under the same finite power of T . (2) If K is the limit of any convergent sequence of point groups of M under T , and if K contains at least one fixed point under T , then K is a connected set. (Received October 29, 1937.)

409. Professor A. A. Shaw: *A new fragment of Euclid's Elements.* Preliminary report.

The library of the University of Pavia, Italy, has a fragment of Euclid's *Elements*, Book I, in ancient Armenian, which was published, without comment, in "Pazmaveb," vol. 42 (1884), Venice, Italy. The manuscript begins with Definition 19 and ends with Proposition III, and has no date. When compared with the two standard editions of Heiberg and Simson, certain omissions and additions are discovered. The nature of these variations seems to indicate either (a) that the translator used an edition of Euclid *different* from Heiberg and Theonine recensions, or (b) that he used a pre-euclidean "Elements." Furthermore, close study of the style and diction leads the present writer to believe that the translation may be as early as the fifth century. (Received October 29, 1937.)

410. Professor Glenn James: *On the first case of Fermat's Last Theorem.*

In a previous paper limits were established for the parameters x, y, z , of the Fermat equation, $x^n + y^n = z^n$, beneath which this equation can have no solution. For the cases $n=3$ and $n=14,000$ these limits were $21,147+$ and $10^{112,300}+$, respectively (14,000 is the limit for n beneath which the equation is known to have no solution). In the present paper these limits are raised to $26,855+$ and $(10/7) 10^{112,300}$, the general limit being raised from $n(2n^2+1)^n$ to $(1/(2^{1/n}-1)-1)$ times that limit. (Received October 29, 1937.)

411. Professor E. T. Bell: *Iterated exponential numbers.*

This twofold infinity of positive integers $\xi_n^{(m)}$, ($m, n=0, 1, \dots$), is defined by $G_0(x) = e^x$, $G_{m+1}(x) = \exp [G_m(x) - 1]$, $G_m(x) = \sum_{n=0}^{\infty} \xi_n^{(m)} x^n / n!$. The $\xi_n^{(1)}$ have been investigated by J. Touchard and the writer. The numbers $\xi_n^{(m)}$ have many remarkable arithmetical properties, including a double periodicity with respect

to a prime modulus p . A specimen of the congruences is $\xi_n^{(m)} \equiv (m+r, m) \pmod{p}$, where $n = pr$, and $(,)$ is a binomial coefficient. The numbers are closely connected with the Stirling numbers of both kinds, but their arithmetical properties are most readily obtained otherwise. For $m = 1$, the numbers are known in combinatorial analysis and in the theory of partitions; for $m > 1$ they do not seem to have been considered before. (Received October 30, 1937.)

412. Mr. H. A. Arnold: *Differentiation in linear, semi-ordered spaces.*

Properties of the Fréchet and the Gateaux differentials are studied relative to abstract semi-ordered spaces as postulated and treated by Kantorovitch. Implicit function theorems are studied from the purely topological standpoint and also later by the aid of the properties of differentials. Methods are indicated for attacking the problems of the existence theorems of differential equations in abstract semi-ordered spaces. (Received October 30, 1937.)

413. Professor A. D. Michal: *General Riemannian spaces with abstract coordinates and constant curvature.*

In this paper, a definition of "Riemannian" curvature is given for the author's general Riemannian differential geometry with Banach coordinates. In case the Banach space is a finite dimensional arithmetic space the definition is equivalent to the one of classical Riemannian geometry. Schur's well known theorem on spaces of constant Riemannian curvature is generalized. Conformally flat and projectively flat general Riemannian spaces with Banach coordinates are then studied and their theory is used in the study of general Riemannian spaces with Banach coordinates and constant "Riemannian" curvature. (Received October 30, 1937.)

414. Dr. A. H. Diamond: *On the elliptic quintic curve in space of four dimensions.*

The author obtains certain properties of the elliptic quintic curve in space of four dimensions by considering correspondences of points on the curve. Through any real point P of the curve pass four primes which osculate the curve at four real points. Quadruplets of points are thus determined on the curve such that the osculating primes at the points of a quadruplet intersect at a point on the curve. A linear construction of the curve is given by means of three such quadruplets. These twelve points determine four groups of sixteen trisecant planes. The curve is shown to be the intersection of four cubic ruled surfaces which consist of the four systems of secants of the curve which intersect the four groups of trisecant planes. (Received November 1, 1937.)

415. Professor Max Zorn: *Topological studies in the theory of analytic functions.* Preliminary report.

It is shown that many essential features of the theory of analytic functions of one complex variable are derivable from a small set of postulates. These postulates are strictly topological, and it is conjectured that they imply the

complete theory of analytic functions on Riemann surfaces. (Received November 1, 1937.)

416. Professor Morgan Ward: *Structure residuation.*

Given any two elements A and B of an arbitrary structure Σ , the author defines the residual of B with respect to A relative to cross-cut as an element $R = A : B$ of Σ with the properties $A \supset [A : B, B]$; $A \supset [X, B]$ implies $A : B \supset X$. Every structure closed with respect to such an operation or its dual must be distributive, and any distributive structure with a chain condition is closed with respect to residuation. If $B \supset A$, the quotient A/B equals the residual $A : B$. The results of a paper, to appear in December in the Duke Mathematical Journal, on residuation in structures over which a multiplication is defined all apply to the present case on identifying multiplication with cross-cut. In particular the two "distributive laws" $[A, B] : C = [A : C, B : C]$, $A : [B, C] = [A : B, A : C]$ always hold. Interesting results ensue if the additional distributive law $A : [B, C] = [A : B, A : C]$ holds. For a structure with descending chain condition, the powers of any irreducible element form an ordered structure. If we define the "kernel" of any element of the structure as the cross-cut of the highest powers of all irreducible elements dividing it, then all elements with the same kernel form an ordered structure. These facts allow an extension of the fundamental theorem of arithmetic to such structures. Every finite structure closed with respect to residuation is isomorphic to a set of positive integers closed under the operations of L. C. M. and G. C. D. Application is made of this result to studying the free distributive structure based on a finite number of elements. (Received November 1, 1937.)

417. Professor P. H. Daus: *Collineations and central projections.*

This paper discusses the collineation between two planes determined by four pairs of corresponding elements. It proves that the collineation may be established by means of three central projections and gives two different geometric constructions for obtaining such a chain of central projections. (Received October 20, 1937.)

418. Dr. R. M. Robinson: *Note on convex regions on the sphere.*

If G is any convex region on the sphere, and G_0 the diametrically opposite region, then there is on the sphere a circle of radius $\arctan(3^{1/2}/2)$, whose interior is either entirely in G or entirely exterior to G and G_0 . There is not always a larger circle with this property. (Received October 28, 1937.)

419. Professor E. T. Bell: *Lagrange and Wilson theorems for the generalized Stirling numbers.*

The generalized Stirling number $S_r^{(m)}(n-1)$ is the coefficient of x^{n-r-1} in $\prod_{a=1}^{n-1} (x+a^m)$. A complete discussion of the congruence properties of these numbers with respect to a prime modulus is given. The case $m=1$ is Lagrange's identical congruence; the case $m=2$ was disposed of by Glaisher, whose method, different from the present one, would be difficult to extend to $m > 2$. (Received October 30, 1937.)

420. Professor E. T. Bell: *Notes on denumerants.*

If n, a_1, \dots, a_r are given integers, $n \geq 0, a_1 > 0, \dots, a_r > 0$, the number $A(n; a_1, \dots, a_r)$ of sets of non-negative integer solutions of $a_1x_1 + \dots + a_rx_r = n$ is the denumerant $A(n)$. The characteristic functional equation of $A(n)$ is derived and used to generalize and simplify theorems on denumerants due to Fergola and Sardi. Also a reciprocity between any denumerant $A(n)$ as above and $A(n; b_1, \dots, b_s)$ is exhibited. This reciprocity also characterizes denumerants. Finally the partition sums of Jacobi, Sylvester, Catalan, and others are generalized, and the generalizations are connected with recurring series. (Received October 30, 1937.)

421. Mr. H. A. Arnold: *Riemannian tensor analysis in abstract spaces.*

A linear form whose values are covariant vectors is taken as a fundamental metric. It is required to be self-adjoint with respect to the contraction operation. The effect of regarding this linear form as the interspace product of an object $g(x)$ in one Banach space with an object $\xi(x)$ in another Banach space is studied with a view to obtaining an abstraction and generalization of the properties of tensors and vectors. (Received October 30, 1937.)

422. Professor A. D. Michal: *Infinitely dimensional Riemannian differential geometries with constant curvature.*

In this paper a special study is made of those geometries introduced in a previous paper (abstract 43-11-413) for which the Banach space is taken to be either a space of continuous functions or a Hilbert space. (Received October 30, 1937.)

423. Dr. Erich Rothe: *Topological proofs of uniqueness theorems in the theory of differential and integral equations.*

Let $y=f(x)$ be a uniform continuous representation defined for all points x of a sphere V of the n -dimensional euclidean space. Then the "order of a point y with respect to the image of the boundary of V " is a well known topological concept; if the equation $y_0=f(x)$ has but one solution in V , the order equals ± 1 . The converse is true only under additional conditions, for instance if the "indices" of the presumable solutions x all have the same sign. It is shown that for a certain class of representations the same consideration holds in Banach space and may be used for the proof of uniqueness theorems concerning differential and integral equations. Proofs of uniqueness theorems based on other topological ideas were first given by R. Caccioppoli (1931) and later on by G. Scorza (1935, 1936). (Received November 1, 1937.)

424. Mr. Philip Hartman: *Mean motions of almost periodic functions and the Riemann zeta function.*

It is proved that every uniformly almost periodic function with linearly independent exponents has a mean motion; a result recently proved (Hartman, van Kampen and Wintner, *Mean motions and distribution functions*, American Journal of Mathematics, vol. 59 (1937), pp. 261-269) only with the exclusion

of certain limiting cases and by using the ergodic theorem of Birkhoff. The present general treatment is based on an application of the Jessen function of analytic almost periodic functions (Jessen, *Über die Nullstellen einer analytischen fastperiodischen Funktion*, *Mathematische Annalen*, vol. 108 (1933), pp. 485–516) and on the other hand, the direct estimate of Fourier transforms in case of convolutions (Wintner, *Upon a statistical method in the theory of diophantine approximations*, *American Journal of Mathematics*, vol. 55 (1933), pp. 309–331). The method can be extended so as to apply, without the assumption of Riemann's hypothesis, both for $\zeta(s)$ and $\log \zeta(s)$ for every fixed $\sigma > 1/2$. (Received October 30, 1937.)

425. Dr. Ruth R. Struik: *The place of Euclid's tenth book in ancient and modern science.*

The tenth book of Euclid, edited in a popular edition by Enriques and the author of this paper, is presented as a unique masterpiece of research of Euclid's time compared to the other twelve books (except perhaps the fifth), which are compendium schoolbooks rather than original treatises. The whole of the thirteen books are considered a "cours d'analyse" of an "academicien" of more than 2,000 years ago, teaching at one of the many primitive colleges of Plato's and Archimedes' time. (Received October 25, 1937.)

426. Dr. Ruth R. Struik: *A problem connected with the "axiomatic of affine geometry."*

The problem is: What is the form of an infinitesimal operator in the sense of Lie which is, microcosmically, an analogy to the affine reflection? The affine reflexion is the transformation of the plane so that all points are mirrored on a straight line s , the mirror being in the direction of a given vector p . It is similar to a reflexion in ordinary euclidean geometry, only the rays are not orthogonal to s , but make an arbitrary angle with s . (Received October 25, 1937.)

427. Dr. O. E. Lancaster: *Some results concerning the behavior at infinity of real continuous solutions of algebraic difference equations.*

This paper treats of the rate of increase of real continuous solutions of algebraic difference equations; that is, of equations of the form $f[x, y(x), y(x+1), \dots, y(x+m)] = 0$, where f is a polynomial in its arguments $x, y(x), y(x+1), \dots, y(x+m)$ with rational coefficients. The following results are obtained: First, a solution of a first order algebraic difference equation cannot equal or exceed $e_2[xl_n(x)]$ for all $x > x_0(n)$, where $e_2(x)$ denotes the second iterate of the exponential, $l_n(x)$ denotes the n th iterate of the logarithm, and n is any integer. Second, $ce_2[xl_n(x)]$ cannot be a solution of any algebraic difference equation. Third, if $y(x)$ is a solution of an algebraic difference equation of the m th order, and if the increase of $y(x)$ satisfies certain very restrictive regularity conditions, then $|y(x)| < e_2[xl_n(x)]$ for $x > x_0(n)$. Fourth, unless regularity conditions are imposed upon the solutions there is no upper bound for the rate of increase of the solutions of all algebraic difference equations of higher than

first order. This last result is a direct analogue of the results obtained by Vijayaraghavan for algebraic difference equations (*Comptes Rendus, Paris*, vol. 194 (1932), pp. 827-829). (Received November 2, 1937.)

428. Professor E. G. Olds: *A moment-generating function which is useful in solving certain matching problems.*

A given target deck consists of n/s different sets of s cards, those in each set being alike. A second deck, identical with the given deck, is thoroughly mixed and the order compared with that of the target deck. This paper is concerned with the moments of the theoretical distribution of numbers of correct matchings. A moment-generating function is derived and the values of the first four moments are obtained. The third and fourth moments are, respectively, $[s(n-s)(n-2s)]/[(n-1)(n-2)]$ and $\{s(n-s)[(n-2s)(n-3s)(3s+1) + (s-1)(12ns-n-18s^2-6s)]\}/[(n-1)(n-2)(n-3)]$ (except for certain trivial cases when $n \leq 3$). The values of the first two moments, obtained in another way, were communicated by Greenwood (abstract 43-9-331). Results for the special case $s=1$ were given by Chapman (*American Journal of Psychology*, vol. 46, p. 294). (Received October 26, 1937.)

429. Professor V. C. Poor: *The Cauchy singular integral and circulation functions.*

In this paper the principal value of the singular integral of Cauchy is expressed in explicit form; so also are the principal values of this integral. Necessary and sufficient conditions are obtained for a function restricted by a Lipschitz condition to be a *circulation function* in a region bounded by a simple Jordan curve. Similar results are obtained for the exterior region. (Received October 29, 1937.)

430. Professor V. C. Poor: *The analog to the Cauchy integral and circulation functions of the second kind.*

The analog to the Cauchy integral is found and studied. Incidentally the analog to the Pompeii theorem is deduced. It is then shown how a circulation function of the second kind may be determined for a region from its boundary values. Both the interior and exterior cases are considered. (Received October 29, 1937.)

431. Dr. W. E. Sewell: *Note on the Faber coefficients of a continuous function.*

Let C be an analytic Jordan curve in the z -plane. Let $f(z)$ be analytic in C , continuous in \bar{C} , and let $f^{(p)}(z)$ have a modulus of continuity $\omega(\delta)$ on C . Let $\sum_{\nu=0}^{\infty} a_{\nu} P_{\nu}(z)$ be the development of $f(z)$ in the Faber polynomials belonging to C . Then we have $|a_{\nu}| \leq (Mr^{\nu/\nu^p})\omega(\epsilon/\nu)$ where M and ϵ are constants independent of ν , and $1/r$ is the capacity of the set bounded by C . (Received November 1, 1937.)

432. Professor F. C. Smith: *Interrelations among the funda-*

mental solutions of the generalized hypergeometric equation when $p=q+1$. I. Non-logarithmic cases.

In this paper, the relations between the non-logarithmic solutions of the generalized hypergeometric equation about the point $z=0$ and those about the point $z=\infty$ are developed for the case in which $p=q+1$. The results generalize those of Mehlenbacher (see abstract 42-5-169) who developed the relations between the fundamental solutions of the hypergeometric equation, and extend those of Barnes (Proceedings of the London Mathematical Society (2), vol. 5 (1907), pp. 59-116) who obtained the asymptotic developments of the solutions of the generalized hypergeometric equation for those cases in which $p < q+1$. (Received October 23, 1937.)

433. Dr. Olga Taussky: *Analytical methods in hypercomplex systems.*

Let S be a hypercomplex system with respect to the field of real numbers. The set Λ of all elements λ such that $\lim_{n \rightarrow \infty} \lambda^n = 0$ throws light on the algebraic structure of the ring in many ways (see also the author's paper in *Compositio Mathematica*, vol. 3 (1936)). Among other results the following may be of interest: The necessary and sufficient condition that for any two elements λ, λ' of Λ the product $\lambda\lambda'$ also is contained in Λ is that the residue class ring S/R (R the radical of S) is the direct sum of division algebras F_1, F_2, \dots and $F_i F_k = 0$ if $i \neq k$. (Received October 23, 1937.)

434. Mr. W. H. Ingram: *Kirchhoff constraints and linear graphs.*

In a connected system of line segments (network) representing a constrained dynamical system, the author introduces topologically (and requires one vertex to be) a *ground-point* or *ground*; also *Lagrangean connection* between each free vertex and ground; and asserts that all currents (velocities) are circuital (traversing the Lagrangean connections in part). Maxwell's topological axiom for unconstrained systems is obtained as a theorem. External forces e_i on the dynamical system are associated with corresponding line segments but associable with the *chords* of any tree by an application of Blakesley's n -furcation theorem. The cyclomatic number is given by the number of segments minus the number of Lagrangean connections. Potentials associated with the vertices are computed from the Lagrangean multipliers associated with the Lagrangean connections. (Received November 1, 1937.)