INEQUALITIES SATISFIED BY A CERTAIN DEFINITE INTEGRAL

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1. *Introduction.* In this note we solve the following problem. Suppose that

$$0 \leqq a_1 < a_2 < \cdots < a_{2n+1} \leqq 1,$$

$$(1) \qquad f(x) = \frac{(x - a_2)(x - a_4) \cdots (x - a_{2n})}{(x - a_1)(x - a_3) \cdots (x - a_{2n+1})},$$

$$J(t) = \int_0^1 |f(x)|^t dx, \qquad 0 < t < 1.$$

Then what are the best inequalities satisfied by $J(t)$?

We prove the following theorem:

THEOREM A. *If $f(x)$ satisfies* (1) *then*

$$\frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1 - t)\pi^{1/2}} \leqq J(t) \leqq \frac{2^t}{1 - t},$$

with inequality except when

$$f(x) = \frac{1}{x - \frac{1}{2}}, \qquad J(t) = \frac{2^t}{1 - t};$$

$$f(x) = \frac{x - \frac{1}{2}}{x(x - 1)}, \qquad J(t) = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1 - t)\pi^{1/2}}.$$

The integral $J(t)$ occurred in a recent paper by Levinson.† Levinson proved that

$$J(t) < \frac{5}{1 - t},$$

and indeed that

$$\int_0^1 |f(x + iy)|^t < \frac{5}{1 - t}$$

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† Levinson, *On non-harmonic Fourier series*, Annals of Mathematics, (2), vol. 37 (1936), p. 922.

for any real y, and stated without proof a more precise, though still not the best possible, inequality. Here we confine ourselves to the case $y=0$, but our results are the best of their kind. We prove them by two methods, one "real" and one "complex".

2. *A Theorem of Boole.* LEMMA 1. *If $f(x)$ satisfies* (1), *then*

$$\int_{-\infty}^{\infty} F\{f(x)\}dx = \int_{-\infty}^{\infty} F(y)\frac{dy}{y^2} \tag{2}$$

whenever (i) *$F(y)$ is defined for all values of y, and* (ii) *either integral exists as a Lebesgue integral.*

Lemma 1 is essentially the same as a theorem of Boole.*

There are two other definitions of $f(x)$ equivalent to that of §1. In the first place, as we can verify at once by resolving $f(x)$ into partial fractions,

$$f(x) = \sum_{\nu=0}^{n} \frac{\alpha_\nu}{x - a_{2\nu+1}}, \tag{3}$$

where

$$\alpha_\nu > 0, \qquad \sum \alpha_\nu = 1. \tag{4}$$

This is the form which we shall generally use here. Secondly

$$g(x) = \frac{1}{f(x)} = x - \sum_{\nu=0}^{n} \frac{\beta_\nu}{x - a_{2\nu}},$$

where $\beta_\nu > 0$. If we write $1/y$ for y and $G(y)$ for $F(1/y)$, then (2) becomes

$$\int_{-\infty}^{\infty} G\{g(x)\}dx = \int_{-\infty}^{\infty} G(y)dy,$$

which is Boole's formula.

To prove Lemma 1 we observe that, after (3) and (4), the graph of $f(x)$ consists of $n+2$ descending pieces corresponding to the intervals $(-\infty, a_1)$, (a_1, a_3), $\cdots$, (a_{2n+1}, ∞), the corresponding intervals of variation of $f(x)$ being $(0, -\infty)$,

* G. Boole, *On the comparison of transcendents, with certain applications to the theory of definite integrals*, Philosophical Transactions of the Royal Society, vol. 147 (1857), pp. 745–803. See in particular p. 780. Boole's very interesting memoir has been forgotten, and his results have been rediscovered, wholly or in part, by a number of later mathematicians.

$(\infty, -\infty), \cdots, (\infty, 0)$; and that, when x moves from $-\infty$ to ∞, y moves, in all, $n+1$ times over the same range. The line $f(x)=y$ cuts the graph of $f(x)$ in $n+1$ points $x_1, x_2, \cdots, x_{n+1}$; and

$$\int_{-\infty}^{\infty} F(y)dx = \int_{-\infty}^{\infty} F(y)P(y)\frac{dy}{y^2},$$

where

$$P(y) = -y^2\sum_{\nu}\left(\frac{dx}{dy}\right)_{x=x_\nu}$$

We have to prove that*

$$P(y) = 1.$$

It is plain that, if $f(x)=y$, then

$$y\prod_{\nu}(x-a_{2\nu+1}) - \sum_{\nu}\alpha_\nu\prod_{\mu\neq\nu}(x-a_{2\mu+1}) = y\prod_{\nu}(x-x_\nu). \tag{5}$$

Hence, first, equating the coefficients of x^{n-1} and using (4), we have

$$\sum x_\nu - \sum a_{2\nu+1} = \frac{1}{y}. \tag{6}$$

Next, (6) is an identity in y when $x_\nu(y)$ is substituted for x_ν. Hence, differentiating this, we obtain

$$\sum\frac{dx_\nu}{dy} = -\frac{1}{y^2}.$$

It follows that $P(y)=1$.

3. *The Underlying Identity.* In what follows it is convenient to symmetrize our analysis about the origin, which we can do by writing $x-\frac{1}{2}$ for x. We have then

$$J(t) = \int_{-1/2}^{1/2} |f(x)|^t dx,\ f(x) = \sum\frac{\alpha_\nu}{x-a_{2\nu-1}},\ \alpha_\nu>0,\ \sum\alpha_\nu = 1, \tag{7}$$

and

$$-\tfrac{1}{2} \leqq a_1 < a_2 < \cdots < a_{2n+1} \leqq \tfrac{1}{2}. \tag{8}$$

* We are indebted to Professor Bohnenblust for a simplification of the proof.

LEMMA 2. *If $f(x)$ satisfies* (7) *and* (8), *then*

$$J(t) = \frac{2^t}{1-t} - \int_{1/2}^{\infty}\left\{|f(x)|^t + |f(-x)|^t - \frac{2}{x^t}\right\}dx. \tag{9}$$

Suppose that ϵ is small and positive and that ξ and η are the largest and smallest roots of $f(x)=\epsilon$ and $f(x)=-\epsilon$ respectively. Then $\xi>\frac{1}{2}$ and $\eta<-\frac{1}{2}$. Also

$$\frac{1}{\xi+\frac{1}{2}} \leqq \sum \frac{\alpha_\nu}{\xi - a_{2\nu+1}} = \epsilon \leqq \frac{1}{\xi-\frac{1}{2}},$$

and so

$$\frac{1}{\epsilon} - \frac{1}{2} \leqq \xi \leqq \frac{1}{\epsilon} + \frac{1}{2},$$

$$\xi = \frac{1}{\epsilon} + O(1), \tag{10}$$

where the O refers to the limit process $\epsilon \to 0$. Similarly

$$\eta = -\frac{1}{\epsilon} + O(1). \tag{11}$$

Define f_ϵ by the relations

$$f_\epsilon = f, \quad (|f| \geqq \epsilon); \qquad f_\epsilon = 0, \quad (|f| < \epsilon).$$

Then, by Lemma 1,

$$\int_{-\infty}^{\infty} |f_\epsilon|^t dx = 2\int_{\epsilon}^{\infty} |y|^{t-2}dy = \frac{2\epsilon^{t-1}}{1-t}.$$

Hence

$$\begin{aligned} J(t) &= \int_{-1/2}^{1/2} |f|^t dx = \lim_{\epsilon\to 0}\int_{-1/2}^{1/2} |f_\epsilon|^t dx \\ &= \lim_{\epsilon\to 0}\left\{\frac{2\epsilon^{t-1}}{1-t} - \left(\int_{1/2}^{\xi} |f|^t dx + \int_{\eta}^{-1/2} |f|^t dx\right)\right\}. \end{aligned} \tag{12}$$

Now

$$f(x) = x^{-1} + O(x^{-2}), \qquad |f(x)|^t = |x|^{-t} + O(|x|^{-t-1})$$

for large x. Hence, by (10),

$$\int_{1/\epsilon}^{\xi} |f|^t dx = \frac{1}{1-t}\left\{\left(\frac{1}{\epsilon} + O(1)\right)^{1-t} - \left(\frac{1}{\epsilon}\right)^{1-t}\right\} + O(\epsilon^t)$$
$$= O(\epsilon^t),$$

and we may replace ξ by $1/\epsilon$ in (12). Similarly we may replace η by $-1/\epsilon$. Hence

$$J(t) = \lim_{\epsilon\to 0}\left\{\frac{2\epsilon^{t-1}}{1-t} - \int_{1/2}^{1/\epsilon}\{|f(x)|^t + |f(-x)|^t\}dx\right\}$$
$$= \lim_{\epsilon\to 0}\left\{\frac{2\epsilon^{t-1}}{1-t} - 2\int_{1/2}^{1/\epsilon}\frac{dx}{x^t}\right.$$
$$\left. - \int_{1/2}^{1/\epsilon}\left\{|f(x)|^t + |f(-x)|^t - \frac{2}{x^t}\right\}dx\right\},$$

which is (9).

4. *A Lemma.* LEMMA 3. *If* $|x| > \frac{1}{2}$ *then*

$$\phi(x) = |f(x)|^t + |f(-x)|^t$$

is (for every x) least and greatest when $f(x)$ is $1/x$ and $x/(x^2-\frac{1}{4})$ respectively.

We may suppose $x > \frac{1}{2}$. We consider the pole A of $f(x)$ nearest to an end of $(-\frac{1}{2}, \frac{1}{2})$. If we suppose, for example, that $A > 0$, then $A = a_{2n+1}$. If

$$\xi = \frac{1}{x-a}, \quad \xi' = \frac{1}{x+a}, \quad \Xi = \frac{1}{x-A}, \quad \Xi' = \frac{1}{x+A},$$

then all these numbers are positive and

$$\frac{\Xi}{\Xi'} > \frac{\xi}{\xi'} \geqq 1 \tag{13}$$

for any pole a other than A. If

$$\psi(A) = \phi(x) = |f(x)|^t + |f(-x)|^t$$
$$= \left(\sum\frac{\alpha}{x-a}\right)^t + \left(\sum\frac{\alpha}{x+a}\right)^t,$$

then

$$\frac{1}{t}\frac{d\psi(A)}{dA} = |f(x)|^{t-1}\frac{\mathrm{A}}{(x-A)^2} - |f(-x)|^{t-1}\frac{\mathrm{A}}{(x+A)^2},$$

where A is the α corresponding to A. This will be positive if

$$\left(\frac{\Xi}{\Xi'}\right)^2 > \left(\frac{\sum \alpha\xi}{\sum \alpha\xi'}\right)^{1-t},$$

and this is true on account of (13).

Hence we decrease $\phi(x)$ by moving A to the left, to the next pole, or to the origin if there is no other positive pole. Similarly, if A were negative, we should decrease $\phi(x)$ by moving A to the right. It follows by repetition of the argument that $\phi(x)$ is least when all the a's coincide at the origin, and $f(x) = 1/x$.

Similarly $\phi(x)$ is greatest when all the a's are at one of the ends of $(-\frac{1}{2}, \frac{1}{2})$. In this case

$$f(x) = \frac{\alpha}{x-\frac{1}{2}} + \frac{1-\alpha}{x+\frac{1}{2}} = \frac{x-\beta}{x^2-\frac{1}{4}},$$

where $\beta = \alpha - \frac{1}{2}$, $0 \leqq \alpha \leqq 1$, $|\beta| \leqq \frac{1}{2}$. Finally

$$|x-\beta|^t + |x+\beta|^t < 2|x|^t$$

if $|x| > \frac{1}{2}$, $\beta \neq 0$, so that the true maximum of $\phi(x)$ occurs when

$$f(x) = \frac{x}{x^2-\frac{1}{4}}.$$

5. *Proof of the Inequalities*. We can now prove the theorem. We take the interval as $(-\frac{1}{2}, \frac{1}{2})$, so that the two critical functions are

$$f_1(x) = \frac{1}{x}, \qquad f_2(x) = \frac{x}{x^2-\frac{1}{4}}.$$

It follows from (9) and Lemma 3 that

$$J(t) \leqq \frac{2^t}{1-t},$$

with inequality unless $f = f_1$. Also

$$\int_{-1/2}^{1/2} |f|^t dx - \int_{-1/2}^{1/2} |f_2|^t dx$$
$$= \int_{1/2}^{\infty} (|f_2(x)|^t + |f_2(-x)|^t - |f(x)|^t - |f(-x)|^t) dx,$$

by (9), and the last integral is positive, by Lemma 3, unless $f = f_2$. Finally

$$\int_{-1/2}^{1/2} |f_2|^t dx = \int_{-1/2}^{1/2} \left| \frac{x}{x^2 - \frac{1}{4}} \right|^t dx = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1 - t)\pi^{1/2}},$$

by an elementary calculation.

6. *Alternative Proof of the Underlying Identity.* There is another proof of (9) by complex integration. We integrate

$$\int \left\{ (f(x))^t - \frac{1}{x^t} \right\} dx$$

around a contour C composed of (i) small semicircles of radius ρ, above the real axis, around the singularities a_k and 0, (ii) a large semicircle of radius R, above the real axis, around 0, and (iii) the parts of the real axis between these semicircles. We suppose

$$(f(x))^t > 0, \qquad x^t > 0$$

for large positive x, and make $\rho \to 0$ and $R \to \infty$ in the usual manner. Then $(f(x))^t$ is positive along

$$(a_1, a_2)\ (a_3, a_4), \cdots, (a_{2n+1}, \infty)$$

and has the argument of $e^{-t\pi i}$ on the rest of the axis, while x^t is positive for $x > 0$ and has the argument of $e^{-t\pi i}$ for $x < 0$. We thus obtain

$$I_1(t) + e^{-t\pi i} I_2(t) = 0, \tag{14}$$

where

$$I_1(t) = \left(\int_{a_1}^{a_2} + \int_{a_3}^{a_4} + \cdots + \int_{a_{2n-1}}^{a_{2n}} + \int_{a_{2n+1}}^{1/2} \right) |f(x)|^t dx$$
$$- \int_0^{1/2} \frac{dx}{|x|^t} + \int_{1/2}^{\infty} \left(|f(x)|^t - \frac{1}{|x|^t} \right) dx,$$

$$I_2(t) = \left(\int_{-1/2}^{a_1} + \int_{a_2}^{a_3} + \cdots + \int_{a_{2n}}^{a_{2n+1}}\right) | f(x) |^t dx$$

$$- \int_{-1/2}^{0} \frac{dx}{| x |^t} + \int_{-\infty}^{-1/2} \left(| f(x) |^t - \frac{1}{| x |^t}\right) dx.$$

If we equate imaginary parts in (14) we obtain

$$\left(\int_{-1/2}^{a_1} + \int_{a_2}^{a_3} + \cdots + \int_{a_{2n}}^{a_{2n+1}}\right) | f(x) |^t dx$$

$$= \frac{2^{t-1}}{1 - t} - \int_{1/2}^{\infty} \left(| f(- x) |^t - \frac{1}{| x |^t}\right) dx;$$

and if we multiply by $e^{t\pi i}$, and equate imaginary parts, we obtain

$$\left(\int_{a_1}^{a_2} + \int_{a_3}^{a_4} + \cdots + \int_{a_{2n+1}}^{1/2}\right) |f(x) |^t dx$$

$$= \frac{2^{t-1}}{1 - t} - \int_{1/2}^{\infty} \left(| f(x) |^t - \frac{1}{| x |^t}\right) dx.$$

Finally (9) follows by addition.

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