FORD ON ASYMPTOTIC DEVELOPMENTS

The Asymptotic Developments of Functions Defined by Maclaurin Series. By Walter B. Ford. University of Michigan Studies, Scientific Series, vol. XI. University of Michigan Press, Ann Arbor, 1936. viii+143 pp.

The subjects of mathematical monographs of the present day are all too often intelligible only to the initiates of their immediate fields. By and large the problems which are enunciable in few and simple terms, and which yet are neither classical nor beyond the reach of analysis, are rare. The problem to which the present book is devoted is such a one. It is, in brief, the following: A function of the complex variable being given through the medium of its Maclaurin series, say

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

to determine its nature and structure in the vicinity of the point $z=\infty$. This problem, though a relatively narrow one to be sure, is one of primary importance, not only in the general field of the theory of functions of a complex variable, but also, as the book at hand amply proves, in the theory of differential equations of the Fuchsian type.

The problem is one of very substantial and inherent difficulty, and much research must needs be done before it will have been solved with any show of completeness. The inroad upon it which is here presented has led to excellent results. In its main outlines the method of attack is the familiar one through the calculus of residues, and the analysis centers chiefly upon the evaluation or appraisal of contour integrals. It is basically assumed that the Maclaurin coefficients c_n depend upon n in the manner $c_n = g(n)$, where $g(w)$, regarded as a function of the complex variable w, fulfills certain conditions of analyticity and order of magnitude in appropriate regions of the w plane. As one would expect, the description of $f(z)$ in the region about $z=\infty$ in general calls for the use of asymptotic representations. Since the domains of validity of such representations are always restricted, the determination of both the domains and the representations valid within them is of necessity carried out. This is done in particular for the case in which c_n is any rational function of n, for the case of functions of so-called exponential type, those in which $c_n \sim h/\Gamma(n+p)$ with h and p any complex constants, and for the functions which are designated as of the Bessel type, those in which

$$c_n \sim \frac{h}{\Gamma(n+p_1)\Gamma(n+p_2)}.$$

As an illustration of the theory of the latter, the asymptotic forms of the Bessel functions themselves are obtained, incidentally without any use being made of their differential equation.

These general function theoretic deductions occupy the first seven chapters of the book. The material is excellently organized and is presented with all the clarity it permits. The earlier chapters are to be regarded as largely expository, though not entirely so, the principal source of material being the extensive researches of E. W. Barnes. The later chapters present results of the author's own research which in the main have not been published heretofore.

Chapter 8, the final chapter of the book, will be found quite disjunct from the earlier work. The focus of the investigation is transferred here from the field of the theory of functions to the field of Fuchsian differential equations, specifically to the differential equation

$$z^2 p_0(z) \frac{d^2y}{dz^2} + z p_1(z) \frac{dy}{dz} + p_2(z) y = 0,$$

in which the functions $p_j(z)$, $j=0, 1, 2$, are quadratic polynomials, subject to the condition that the finite singular points of the differential equation be all regular. The classical general theory of this equation establishes the existence, in connection with each singular point, of a pair of solutions with specific indices. It does not, however, supply any method for relating the solutions associated with different singular points, and such relations have, in fact, been derived heretofore only for special differential equations, notably the hypergeometric equation. A theory of considerable generality bearing upon this important point is here developed. If the attention is fixed upon a solution relative to the point $z=0$, its series coefficients $c_n = g(n)$ are, of course, given by a certain recursion formula which is fixed by the differential equation. This formula is here regarded as a difference equation of which the function $g(w)$ is a solution. For the study of $g(w)$ the modern theory of difference equations is drawn upon to a considerable extent. Once its functional structure has been determined the results of the earlier chapters become available, and from them the representations of the solution in question in the vicinity of the point $z=\infty$ are obtained. A considerable differentiation into cases is necessary and is carried out, the form of the solutions depending, as is familiar, largely upon such items as the regularity or irregularity of the point $z=\infty$, whether or not the indices associated with this or that singular point do or do not differ by an integer, and so on. The material of this chapter is again practically all due to the author.

In general appearance and typographical style the book closely resembles the author's well known *Studies on Divergent Series and Summability* which appeared in 1916 in the same series. The book is thorough and serious, and the exposition is good. It is not to be inferred from this, however, that the text reads easily, for, in the author's own words, "the subject inherently calls for complicated forms of expression, and the proofs involve a regrettable amount of heavy analysis." At appropriate points, directions for further research are indicated. This reviewer feels that thanks are due Professor Ford for a significant work.

A list of errata, mainly furnished by the author, is appended.

Page 2, equation (7). Replace (z) by $f(z)$.
Page 12, equation (46). Replace $\sqrt{z}$ by $\sqrt{-z}$.
Page 14, 3d line beneath (51). Omit the words "except those."
Page 17, equation (4). Replace $i(\psi\}-\pi)$ by $i(\psi-\pi)\}$.
Page 22, line 5 in §6. Replace (3) by (5).
Line 6 from bottom. Replace (3) by (5).

Page 23, equation (4). Replace (-1) by $(-1)^n$.
Page 33, 1st line above (19). Replace (11) by (12).
Page 52, relation (61). Replace $\frac{d}{dw} P(w)$ by $w \frac{d}{dw} P(w)$.
Page 68, 7th line of (*b*). Replace $e^z z^{-1p}$ by $e^z z^{1-p}$.
Page 74, equation (3). Replace $\Gamma(z+a_2)/2)$ by $\Gamma(\ (z+a_2)/2)$.
Page 75. Put $\}$ at end of third line.
Page 78, 5th line. Omit = after $\frac{2^{2x-1}}{\sqrt{\pi}}$.
Page 78, 1st line above series expression for $h(t)$. Replace 22 by 25.
Page 80, 5th line below equation (35). Replace arg z by arg t.
Page 93, 4th line below Fig. 9. Replace $\geqq$ by $\leqq$.
Page 105, 14th line. Replace $f_0(x)$ by of $f_0(x)$.
Page 110, 1st line beneath (82). Replace V by γ.
Page 134, 4th line of 2. Replace $R(z)>0$ by $-\pi<\arg z<0$; also replace "section" by "sector."
In 5th line replace $R(z)<0$ by $0<\arg z<\pi$.

RUDOLPH E. LANGER

LEVY AND ROTH ON PROBABILITY

Elements of Probability. By H. Levy and L. Roth. Oxford, Clarendon Press, 1936. i+196 pp.

This book pretends to be "no more than an elementary treatment," and so makes no effort to cover all the many applications of probability, or even that part of statistics which is probability. It is not an elementary text in the sense in which American authors use the term, but it does begin at the beginning, and makes no mathematical demands on the reader beyond a knowledge of the ordinary calculus and some finite differences.

The first five chapters have to do with the usual matters, definitions of probability, arrangements, Bernoulli's theorem, and what the authors occasionally call the normal law, but usually refer to as the Gaussian law, although it is now known that the credit for it does not belong to Gauss. There is in Chapter VI an attempt at a rigorous presentation of an extension of the definition of probability for discrete numbers to continuous distributions; but surely it cannot be made wholly rigorous without more restrictions on the curve than that it be continuous. The net result is that the probability that a point lie on a certain arc is proportional to the length of that arc. It would seem therefore that this result might better be taken as a definition, for, as it is, one gets the impression that the length of an arc has something to do with the number of points it contains, which is not true, and of course is not an inference which the authors would like to have made. In Chapter VIII various forms of probability distributions are considered, including the Poisson distribution and those represented by the use of Hermite's polynomials.

The last chapter (IX) has to do with "scientific induction." Much of this chapter is too condensed for most readers unless they are familiar with the