A SIMPLIFIED SOLUTION OF THE EQUATION $\Delta y(x) = F(x)$*

BY I. M. SHEFFER

We have elsewhere considered, from the point of view of a local solution, the equation

$$(1) \qquad \Delta y(x) \equiv y(x+1) - y(x) = F(x),$$

and certain related equations.† It has now been found possible to simplify the form of the solution of (1) by altering the method. This we propose to show in the present note. Suitably modified, this method may well be expected to apply to other equations.

Let Δ be the difference operator. If t is a parameter, then

$$(2) \qquad \Delta[e^{tx}] = e^{tx}(e^t - 1),$$

so that

$$(3) \qquad e^{tx}(e^t - 1) = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where $P_n(x)$ is the polynomial defined by

$$(4) \qquad P_n(x) = \Delta\left[\frac{x^n}{n!}\right] = \frac{1}{n!}\left\{(x+1)^n - x^n\right\}.$$

Suppose that the coefficient of t^n is multiplied (for every n) by $n!$ on both sides of (3). This yields

$$(5) \qquad \frac{1}{1 - t(x+1)} - \frac{1}{1 - tx} = \sum_0^{\infty} n!P_n(x)t^n.$$

We now transform (5) by replacing t by $1/t$ and dividing through by t:

$$(6) \qquad \frac{1}{t - 1 - x} - \frac{1}{t - x} = \sum_0^{\infty} \frac{n!P_n(x)}{t^{n+1}}.$$

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* Presented to the Society, September 5, 1936.

† *A local solution of the difference equation $\Delta y(x) = F(x)$ and of related equations*, Transactions of this Society, vol. 39 (1936), pp. 345–379.

Our aim is to obtain a P_n-expansion for the single term $1/(t-x)$. This requires that we get rid of one of the terms on the left side of (6), a condition that can be realized in two ways by iteration. This will give us two expansions for $1/(t-x)$, both of which we need.

Let t in (6) be replaced successively by $t-1, t-2, \cdots, t-k$, and the corresponding relations (including (6)) added. This gives

$$\tag{7} \begin{aligned} &\frac{1}{t-(k+1)-x} - \frac{1}{t-x} \\ &\quad = \sum_{n=0}^{\infty} n!P_n(x)\left[\frac{1}{t^{n+1}} + \frac{1}{(t-1)^{n+1}} + \cdots + \frac{1}{(t-k)^{n+1}}\right]. \end{aligned}$$

Likewise, on replacing t in (6) by $t+1, t+2, \cdots, t+k$ and adding (this time not using (6)), we obtain

$$\tag{8} \begin{aligned} &\frac{1}{t-x} - \frac{1}{t+k-x} \\ &\quad = \sum_{0}^{\infty} n!P_n(x)\left[\frac{1}{(t+1)^{n+1}} + \cdots + \frac{1}{(t+k)^{n+1}}\right]. \end{aligned}$$

If k is allowed to become infinite, the suggested (but as yet *formal*) relations are

$$\tag{9} \frac{1}{t-x} \sim -\sum_{n=0}^{\infty} P_n(x)\left\{\frac{1}{t^{n+1}} + \frac{1}{(t-1)^{n+1}} + \cdots\right\} n!,$$

$$\tag{10} \frac{1}{t-x} \sim \sum_{n=0}^{\infty} P_n(x)\left\{\frac{1}{(t+1)^{n+1}} + \frac{1}{(t+2)^{n+1}} + \cdots\right\} n!.$$

The series in the braces converge for $n>0$, but diverge for $n=0$. Fortunately $P_0(x)\equiv 0$; hence the series can start at $n=1$. We can determine the region of convergence of (9) and (10) as follows.

Define functions* $H(t)$ and $K(t)$ by

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* The additional terms $-1/r$, $1/r$ have been inserted in order to secure convergence. The functions H, K are well known in the theory of the Gamma Function, and it is of interest to observe in how natural a manner they arise in the present paper.

(11) $$H(t) = \sum_{r=1}^{\infty}\left[\frac{1}{t+r} - \frac{1}{r}\right], \quad K(t) = \frac{1}{t} + \sum_{r=1}^{\infty}\left[\frac{1}{t-r} + \frac{1}{r}\right].$$

These series converge uniformly in any bounded region whose distance (respectively) from the set of points $-1, -2, -3, \cdots$; $0, 1, 2, \cdots$, is positive. Now

(12) $$\frac{d^nH(t)}{dt^n} = (-1)^n\cdot n!\sum_{r=1}^{\infty}\frac{1}{(t+r)^{n+1}},$$
$$\frac{d^nK(t)}{dt^n} = (-1)^n\cdot n!\sum_{r=0}^{\infty}\frac{1}{(t-r)^{n+1}}.$$

Equations (9) and (10) can therefore be reformulated as

(13) $$\frac{1}{t-x} = -\sum_{0}^{\infty}(-1)^n\cdot K^{(n)}(t)P_n(x),$$

(14) $$\frac{1}{t-x} = \sum_{0}^{\infty}(-1)^nH^{(n)}(t)P_n(x).$$

We now determine if, and for what values, (13) and (14) are valid. Let $M(x, t)$ denote the right hand member of (13). From

(a) $$K(u) = \sum_{0}^{\infty}\frac{(u-t)^n}{n!}K^{(n)}(t),$$

we obtain (using (4)),

(b) $$M(x, t) = K(t-x) - K(t-x-1),$$

and therefore from (11),

(c) $$M(x, t) = \sum_{r=0}^{\infty}\left\{\left[\frac{1}{t-x-r} - \frac{1}{r}\right] - \left[\frac{1}{t-x-1-r} - \frac{1}{r}\right]\right\}$$
$$= \frac{1}{t-x}.$$

That is, (13) is valid whenever the series converges. Now (a) holds provided $t\neq 0, 1, 2, \cdots$, and

$$|u-t| < \min\{|t|, |t-1|, |t-2|, \cdots\}.$$

For (b) we therefore require that*

(i) $|x|$ and $|x+1| < \min\{|t|, |t-1|, |t-2|, \cdots\}$.

* Condition (i) implies the previous condition $t\neq 0. 1, 2, \cdots$.

It follows from (i) that $t-x\neq 0, 1, 2, \cdots$, so that (c) is valid.

To sum up: *Expansion* (13) *is valid for all x and t satisfying condition* (i); and series (13) converges uniformly for x and t in any regions for which (i) holds *uniformly*. (That is, for all x and t in their respective regions, there is a positive number λ such that $\min\{|t|, |t-1|, \cdots\}\geqq\lambda$, and a positive number $\sigma<\lambda$ such that $|x|$ and $|x+1|$ never exceed σ.)

In like manner it can be shown that *expansion* (14) *is valid for all x and t satisfying the condition*

$$\text{(ii)}\quad |x| \text{ and } |x+1| < \min\{|t+1|, |t+2|, |t+3|, \cdots\};$$

and (14) converges uniformly for x and t in any regions for which (ii) holds *uniformly*.

With points 0 and -1 as centers, draw two arcs of radius σ (any number exceeding 1/2). Let γ_1 denote the right hand arc and γ_2 the left. They meet on the line $R(x)=-1/2$. Let $\mathcal{L}$ be the open region enclosed by γ_1 and γ_2. Now let $\mathcal{E}$ be any closed set in $\mathcal{L}$, and restrict x to lie in $\mathcal{E}$. It is readily seen that if t traverses arc γ_2, condition (i) is satisfied uniformly, so that (13) holds uniformly. Similarly, if t traverses γ_1, series (14) converges uniformly.

Let $F(x)$ be any function analytic at $x=-1/2$. We can find a number $\sigma>1/2$ so that if the arcs γ_1, γ_2 are drawn with radius σ, then $F(x)$ is analytic in $\mathcal{L}$ and on its boundary $C=\gamma_1+\gamma_2$. If, then, we substitute for $1/(t-x)$ in Cauchy's integral

$$F(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{(t-x)}\,dt$$

the expansions (13) and (14), we obtain the following theorem.

THEOREM 1. *The function $F(x)$, analytic about $x=-1/2$, has the P_n-expansion*

$$\text{(15)}\qquad F(x) = \sum_{n=0}^{\infty} f_n P_n(x),$$

where

$$\text{(16)}\quad f_n = \frac{(-1)^n}{2\pi i}\int_{\gamma_1} F(t)H^{(n)}(t)dt - \frac{(-1)^n}{2\pi i}\int_{\gamma_2} F(t)K^{(n)}(t)dt,$$

the integrals taken so that the boundary C is traversed in the positive sense. Series (15) *converges uniformly for x in every closed region in* $\mathcal{L}$, *and is therefore valid for all x in* $\mathcal{L}$.

From this we obtain the following result.

THEOREM 2.* *The series*

$$y(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \tag{17}$$

converges in a circle of radius exceeding 1/2, *and in some neighborhood of* $x = -1/2$ *the function* $y(x)$ *is a solution of the equation*

$$\Delta y(x) = F(x). \tag{1}$$

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ON THE SUMMABILITY BY POSITIVE TYPICAL MEANS OF SEQUENCES $\{f(n\theta)\}$†

BY M. S. ROBERTSON

1. *Introduction.* In a recent paper‡ the author required an inequality for the expression

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\sin k\theta| \leqq \frac{1}{\pi}\int_0^{\pi} |\sin\theta|\, d\theta = \frac{2}{\pi}, \tag{1}$$

which apparently is due to T. Gronwall.§ This inequality suggests immediately the question: For what functions $f(\theta)$, defined in the interval $(-\pi, \pi)$, are we permitted to write

$$F(\theta; f) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\theta) \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta? \tag{2}$$

More generally, we may ask: For what functions $f(\theta)$ and sequences $\{a_n\}$ of positive numbers is the following true:

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* See Transactions of this Society, loc. cit., p. 359.

† Presented to the Society, April 11, 1936.

‡ See M. S. Robertson, *On the coefficients of a typically-real function*, this Bulletin, vol. 41 (1935), p. 569.

§ See Transactions of this Society, vol. 13 (1912), pp. 445–468.