ON CERTAIN ARITHMETIC FUNCTIONS OF SEVERAL ARGUMENTS*

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1. *Introduction.* Series of the type

$$\sum_{l,m,n} \beta(l, m, n), \tag{1}$$

summed over all positive l, m, n satisfying the conditions

$$(m, n) = (n, l) = (l, m) = 1, \tag{2}$$

occur in a problem in additive arithmetic. The series (1) is transformed into a series $\sum \gamma(l, m, n)$, now summed over all positive l, m, n, where

$$\gamma(l, m, n) = \sum_{e,f,g=1}^{\infty} \mu(e, f, g)\beta(el, fm, gn).$$

The function $\mu(e, f, g)$ may be defined by

$$\sum \mu(e, f, g) = \begin{cases} 1 & \text{for } (m, n) = (n, l) = (l, m) = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

the summation on the left extending over all $e|l, f|m, g|n$.

In this note we define a class of functions μ satisfying relations of the type (3); the functions generalize, in several directions, the ordinary Möbius μ-functions. We next define and evaluate a class of generalized ϕ-functions; they may be expressed in terms of μ.

2. *The μ-Functions.* For arbitrary positive k, s we define the function $\mu^s(m_1, \cdots, m_k)$ by means of

$$\sum_{e_i|m_i} \mu^s(e_1, \cdots, e_k) = \begin{cases} 1 & \text{for } M^s, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

the k-fold summation on the left extending over all $e_i|m_i$, $(i=1, \cdots, k)$, while M^s is an abbreviation for the $C_{k,s}$ simultaneous conditions

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$$(m_{i_1}, \cdots, m_{i_s}) = 1, \qquad (i_1, \cdots, i_s = 1, \cdots, k;\ i_a \neq i_b).$$

Evidently by means of (4) μ^s may be calculated recursively. The function is symmetric in the k arguments $m_1, \cdots, m_k$.

In the case $s=1$, M^s evidently reduces to $m_i=1$, and thus

$$\mu^1(m_1, \cdots, m_k) = \mu(m_1) \cdots \mu(m_k), \tag{5}$$

where $\mu(e)$ on the right is the ordinary μ-function; for $s>1$, however, no such reduction is in general possible.

From (4) it follows at once that

$$\mu^s(1, 1, \cdots, 1) = 1. \tag{6}$$

In the next place it is not difficult to show that $\mu^s(m_1, \cdots, m_k)$ is *multiplicative* in the k arguments $m_1, \cdots, m_k$. An arithmetic function $f(m_1, \cdots, m_k)$ is multiplicative provided

$$f(m_1, \cdots, m_k) = \prod_p f(p^{e_1}, \cdots, p^{e_k}), \tag{7}$$

where p is a typical prime, and

$$m_i = \prod p^{e_i}, \qquad e_i = e_i(p).$$

Thus the calculation of $\mu^s(m_1, \cdots, m_k)$ is reduced to the calculation of

$$\mu^s(p^{e_1}, \cdots, p^{e_k}), \tag{8}$$

where some of the e_i may be equal to 0. Assume now that some $e_i>1$, say $e_1>1$. Then comparing (4) for

$$p^{e_1}, p^{e_2}, \cdots, p^{e_k} \quad \text{with} \quad p^{e_1-1}, p^{e_2}, \cdots, p^{e_k},$$

leads at once to

$$\mu^s(p^{e_1}, \cdots, p^{e_k}) = 0, \tag{9}$$

if any $e_i>1$. We may therefore suppose in (8) that

$$e_i = 1 \quad \text{or} \quad 0, \qquad (i = 1, \cdots, k).$$

If $e_i=1$ for t values of i, and $e_i=0$ for the remaining $k-t$ values, we may use in place of (8) the simplified notation

$$\mu^s(p^t 1^{k-t}). \tag{10}$$

Again, inspection of the defining equation (4) for the values

$$m_1 = \cdots = m_t = p, \qquad m_{t+1} = \cdots = m_k = 1,$$

shows that the function (10) is independent of k. We may therefore shorten (10) to $\mu^s(p^t)$ or even $\mu^s(t)$ when there is no danger of confusion.

To calculate $\mu^s(p^t)$ we again use (4). Assume first $t<s$. Thus the conditions M^s are surely satisfied. Making use of (6), we show by applying (4) for $t=1, 2, \cdots, t$, that

$$(11) \qquad \mu^s(p^t) = 0 \qquad \text{for} \qquad t = 1, \cdots, s-1.$$

For $t \geqq s$, the conditions M^s are not satisfied. For example, for $t=s$, (4) becomes

$$\mu^s(1) + \mu^s(p^s) = 0,$$

so that $\mu^s(p^s) = -1$. Generally for $t \geqq s$, (4) implies

$$(12) \quad 1 + C_{t,s}\mu^s(p^s) + C_{t,s+1}\mu^s(p^{s+1}) + \cdots + C_{t,t}\mu^s(p^t) = 0.$$

For the moment, put $\mu^s(p^t) = y_t$; then (12) implies

$$(13) \qquad \sum_{i=0}^{t} C_{t,i}y_i = \begin{cases} -1 & \text{for} \quad t = 0, \cdots, s-1, \\ 0 & \text{for} \quad t \geqq s. \end{cases}$$

To solve (13) for y_i, we note that

$$\sum_{t=0}^{w} (-1)^{w-t}C_{w,t}\sum_{i=0}^{t} C_{t,i}y_i$$

$$= \sum_{i=0}^{w} (-1)^{w-i}C_{w,i}y_i\sum_{t=i}^{w} (-1)^{w-t}C_{w-i,t-i} = y_w.$$

Therefore we have

$$y_w = \sum_{t=0}^{s-1} (-1)^{w-t}C_{w,t} = (-1)^{w-s-1}C_{w-1,s-1},$$

as may be proved by an easy induction on s. Recalling the definition of y_w, we see that

$$(14) \qquad \mu^s(p^{s+t}) = (-1)^{t-1}C_{s+t-1,s-1} \qquad \text{for} \qquad t \geqq 0.$$

It is now easy to evaluate $\mu^s(m_1, \cdots, m_k)$ generally. We use (11), (14), and the multiplicative property. Then in the first place, by (9), μ^s vanishes if any s is divisible by the square of a prime. Assume therefore that each m_i is the product of distinct primes p_i. Put

$$m_1 m_2 \cdots m_k = p_1^{t_1} p_2^{t_2} \cdots p_w^{t_w}. \tag{15}$$

Then if any $t_i < s$, it follows from (11) that $\mu^s = 0$. If, however, in (15) each $t_i \geqq s$, then $\mu^s \neq 0$, and is determined by the following formula:

$$\mu^s(m_1, \cdots, m_k) = \prod_{i=1}^{w} (-1)^{t_i - s - 1} C_{t_i - 1, s-1}, \tag{16}$$

which holds generally for all m provided $\mu(m_1) \neq 0, \cdots, \mu(m_k) \neq 0$. Formulas (15) and (16), together with $\mu^s(m_1, \cdots, m_k) = 0$ for $\mu(m_1)\mu(m_2) \cdots \mu(m_k) = 0$, determine μ^s in all cases.

3. *An Application.* By means of the general μ^s, we may transform the series

$$\sum_{M^s} \beta(m_1, \cdots, m_k), \tag{17}$$

summed over all positive m_i satisfying the condition M^s of (4). Now by (4), the series in (17) equals

$$\begin{aligned} &\sum_{(m)=1}^{\infty} \beta(m_1, \cdots, m_k) \sum_{e \mid m} \mu^s(e_1, \cdots, e_k) \\ &= \sum_{(m)=1}^{\infty} \sum_{(e)=1}^{\infty} \mu^s(e_1, \cdots, e_k) \beta(e_1 m_1, \cdots, e_k m_k), \\ &= \sum_{(m)=1}^{\infty} \gamma(m_1, \cdots, m_k), \end{aligned} \tag{18}$$

where

$$\gamma(m_1, \cdots, m_k) = \sum_{(e)=1}^{\infty} \mu^s(e_1, \cdots, e_k) \beta(e_1 m_1, \cdots, e_k m_k). \tag{19}$$

Formulas (18) and (19) effect the transformation.

The example mentioned in the Introduction is the special case $s=2$, $k=3$.

4. *The ϕ-Functions.* For arbitrary positive k, s we define the function $\phi^s(m_1, \cdots, m_k)$ as the number of sets of integers

$$\{e_1, \cdots, e_k\}, \qquad e_i \pmod{m_i},$$

for which W^s holds; W^s is an abbreviation for the $C_{k,s}$ simultaneous conditions

$$(e_{i_1}, \cdots, e_{i_s}, m_{i_1}, \cdots, m_{i_s}) = 1,$$
$$(i_1, \cdots, i_s = 1, \cdots, k; i_a \neq i_b).$$

Clearly ϕ^s is symmetric in the k arguments $m_1, \cdots, m_k$. For $s=1$, W^s reduces to $(e_i, m_i)=1$, so that

$$\phi^1(m_1, \cdots, m_k) = \phi(m_1) \cdots \phi(m_k),$$

where $\phi(m)$ on the right is the ordinary ϕ-function. In the other extreme case, $s=k$, assume $m_1 = \cdots = m_k$; then clearly

$$\phi^k(m, \cdots, m) = \phi_k(m),$$

where $\phi_k(m)$ is Jordan's function. From the definition, it is evident that

$$\phi^s(1, 1, \cdots, 1) = 1.$$

Secondly it is not difficult to show that ϕ^s satisfies (7); in other words, ϕ^s is a multiplicative function of $m_1, \cdots, m_k$. We proceed to calculate

$$\phi^s(p^{e_i}, \cdots, p^{e_k}). \tag{20}$$

If some $e_i > 1$, (20) may be reduced further. Thus, if say $e_1 > 1$, it follows from the definition that

$$\phi^s(p^{e_1}, p^{e_2}, \cdots, p^{e_k}) = p\phi^s(p^{e_1-1}, p^{e_2}, \cdots, p^{e_k}). \tag{21}$$

It is therefore necessary to calculate the function only in the case $e_i = 1$ or 0. Exactly as in §2, if $e_i = 1$ for t values and $=0$ for the remaining $k-t$ values, we replace (20) by the simpler notation

$$\phi^s(p^t 1^{k-t}) = \phi^s(p^t),$$

for here again the function in question is easily seen to be independent of k.

The determination of $\phi^s(p^t)$ involves no difficulty. It follows from the definition that $\phi^s(p^t) = p^t$ for $t < s$. For $t \geqq s$, we may show that

$$\phi^s(p^t) = (p-1)^{t-s+1} \sum_{i=0}^{s-1} C_{t-s+i,i} p^{s-1-i}. \tag{22}$$

Indeed, by the definition,

$$p^t - \phi^s(p^t) = C_{t,t-s}(p-1)^{t-s} + C_{t,t-s-1}(p-1)^{t-s-1} + \cdots,$$

so that

$$\phi^s(p^t) = \sum_{i=0}^{s-1} C_{t,i}(p-1)^{t-i}, \tag{23}$$

which may be identified with (22).

Again, expanding the right member of (23), we have

$$\begin{aligned}\phi^s(p^t) &= \sum_{i=t-s+1}^{t} C_{t,i}\sum_{j=0}^{i}(-1)^{i-j}C_{i,j}p^j \\ &= \sum_{j=0}^{t} C_{t,j}p^{t-j} \sum_{\substack{i=0 \\ i\geqq j-s+1}}^{j} (-1)^i C_{j,i} \\ &= p^t + \sum_{j=s}^{t} C_{t,j}p^{t-j}\sum_{i=0}^{s-1}(-1)^{j-i}C_{j,i} \\ &= p^t + \sum_{j=s}^{t}(-1)^{j-s-1}C_{j-1,s-1}C_{t,j}p^{t-j} \\ &= p^t + \sum_{j=s}^{t} C_{t,j}p^{t-j}\mu^s(p^j) \\ &= \sum_{j=0}^{t} C_{t,j}p^{t-j}\mu^s(p^j),\end{aligned}$$

by (14). Therefore, by (21) and (9),

$$\phi^s(p^{e_1}, \cdots, p^{e_k}) = p^{e_1+\cdots+e_k}\sum_{f_i\leqq e_i}\frac{\mu^s(p^{f_1}, \cdots, p^{f_k})}{p^{f_1+\cdots+f_k}}.$$

Finally, since both ϕ^s and μ^s are multiplicative,

$$\phi^s(m_1, \cdots, m_k) = m_1\cdots m_k\sum_{d_i\mid m_i}\frac{\mu^s(d_1, \cdots, d_k)}{d_1\cdots d_k},$$

and thus ϕ^s is expressed in terms of μ^s.

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