PRODUCTS OF NÖRLUND TRANSFORMATIONS*

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1. *Introduction.* For certain purposes the *symmetric* product of two Nörlund transformations, to be presently defined, has been found to be more convenient than the ordinary product. The aim of this paper is to compare the fields of convergence of these two products. Let M be the ordinary arithmetic mean and P any Nörlund transformation. The principal results obtained in this paper are *first*, that the field of convergence of the ordinary product includes that of the symmetric product; *secondly*, that the converse is in general not true; and *thirdly*, that a necessary and sufficient condition for the equivalence of these products can be stated.

2. *Permutability with M.* A Nörlund transformation [1]† is a special case of that corresponding to a triangular matrix

$$(1) \qquad a_{nk} = \frac{p_{n-k}}{P_n}, \qquad (n \geqq k \geqq 0;\, P_n = p_0 + p_1 + \cdots + p_n),$$

where p_n is a sequence of positive numbers. If

$$(2) \qquad \frac{p_n}{P_n} \to 0,$$

then the matrix a_{nk} is regular. We may assume without loss of generality that $p_0 = 1$. For if $p_0 = 0$, we may take

$$y_{n+1} = \frac{p_0 x_{n+1} + p_1 x_n + \cdots + p_{n+1} x_0}{p_0 + p_1 + \cdots + p_{n+1}} = \frac{p_1 x_n + \cdots + p_{n+1} x_0}{p_1 + p_2 + \cdots + p_{n+1}},$$

which is the result of applying the transformation corresponding to the sequence p_{n+1} to the sequence x_n. If now $p_0 \neq 0$, we may obviously choose $p_0 = 1$.

It has been shown [1], [3], that all Nörlund transformations are consistent. M. Riesz [2] has given necessary and sufficient

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† Such bold-faced numerals refer to the bibliography at the end of this paper.

conditions for the equivalence of two Nörlund transformations. It has also been proved [**3**] that Abel's definition includes all the definitions of Nörlund type.

An important special case of the Nörlund matrix is C_r, the Cesàro matrix of order r. This is given by any one of the formulas

$$(3) \qquad p_n = C_{n+r-1,n}, \qquad P_n = C_{n+r,n}, \qquad a_{nk} = \frac{C_{n-k+r-1,n}}{C_{n+r,n}}.$$

It is well known [**4**] that the matrix C_r is permutable with M. We now state that the Cesàro means are the *only* Nörlund transformations which are permutable with M.

THEOREM 1. *A necessary and sufficient condition that a Nörlund matrix be permutable with M is that it be a Cesàro matrix.*

To prove this we make use of a theorem of Hurwitz and Silverman [**4**] to the effect that the triangular matrix is permutable with M when and only when the elements a_{nk} are given in terms of those in the principal diagonal by the formula

$$(4) \qquad a_{nk} = \sum_{h=k}^{n} (-1)^{h-k} C_{n,h} C_{h,k} a_{hh}.$$

We have then in particular, for $k = n-1$,

$$a_{n,n-1} = n(a_{n-1,n-1} - a_{nn})$$

or

$$\frac{p_1}{P_n} = n\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right),$$

which reduces to

$$p_n = \frac{n + p_1 - 1}{n} p_{n-1}.$$

Taking $p_0 = 1$, $p_1 = r$, we find $p_2 = C_{r+1,2}$, and, by mathematical induction, $p_n = C_{n+r-1,n}$. This proves the necessity of the condition. That the condition is sufficient is well known [**4**].

3. *The Symmetric Product.* We now introduce the notation

$$
\begin{aligned}
(ab)_n &= (ba)_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0, \\
(abc)_n &= ((ab)c)_n = (a(bc))_n, \qquad\qquad (5)\\
(a1)_n &= A_n = a_0 + a_1 + \cdots + a_n.
\end{aligned}
$$

In this notation, the Nörlund transformation may be written in the form

$$y_n = \frac{(px)_n}{(p1)_n} = \frac{(px)_n}{P_n}.$$

We shall now define [1] by the side of the ordinary product of two Nörlund transformations, PQ, the *symmetric product* of the two transformations, symbolically designated by $P \circ Q$; namely, the transformation

$$y_n = \frac{(pqx)_n}{(pq1)_n}.$$

It is clear that $P \circ Q = Q \circ P$, and that if P and Q are regular so is their symmetric product. A generalization* of Cesàro's well known theorem on the product of two series may then be stated as follows:

THEOREM 2. *If the series $\sum u_n$ is summable to u by the Nörlund definitions P, and $\sum v_n$ is summable to v by the definition Q, then the Cauchy product $\sum (uv)_n$ is summable to uv by the definition $P \circ Q \circ M$.*

4. *Relationship between the Two Products.* We now wish to inquire into the relationship between the ordinary product and the symmetric product of two Nörlund transformations. That they are not always equivalent will be seen from the following theorems.

THEOREM 3. *The ordinary product MP sums to the proper value every sequence summed by the symmetric product $M \circ P$.*

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* Theorem 2 was given in a note by the author at the International Congress at Bologna in 1928, but was never published by him. A proof has been given by Florence Mears in a recent number of this Bulletin [5].

THEOREM 4. *There exists a sequence x_n and a Nörlund definition P such that the sequence x_n is summable by the ordinary product MP but not by the symmetric product $M \circ P$.*

THEOREM 5. *A necessary and sufficient condition that the two products MP and $M \circ P$ be equivalent is*

$$(n+1)P_n < K(P1)_n, \tag{6}$$

where K is a constant independent of n.

5. *Proof of Theorem* 3. Let $A = MP$ and $B = M \circ P$. Then we wish to show that the transformation AB^{-1} is regular. The matrix a_{nk} corresponding to the transformation A is given by the formula

$$a_{nk} = \frac{1}{n+1} \sum_{h=k}^{n} \frac{p_{h-k}}{P_h}. \tag{7}$$

The matrix corresponding to B is given by the equation

$$b_{nk} = \frac{P_{n-k}}{P_n}, \tag{8}$$

so that the transformation B may be written in the form

$$z_n = (P1)_n y_n = (Px)_n.$$

Now we know that

$$\sum z_n t^n = \sum (Px)_n t^n = \sum P_n t^n \cdot \sum x_n t^n.$$

Let us define $\bar{p}_n$ and $\bar{P}_n$ by the formal relations

$$\sum p_n t^n \cdot \sum \bar{p}_n t^n = 1, \qquad \sum P_n t^n \cdot \sum \bar{P}_n t^n = 1, \tag{9}$$

so that

$$(p\bar{p})_0 = p_0 \bar{p}_0 = 1, \qquad (p\bar{p})_n = 0, \qquad (n > 0); \tag{10}$$

and since

$$\sum p_n t^n = (1-t) \sum P_n t^n,$$

we have

$$\bar{P}_0 = \bar{p}_0, \qquad \bar{P}_m = \bar{p}_m - \bar{p}_{m-1}, \qquad (m > 0). \tag{11}$$

We now find

$$\sum x_n t^n = \sum \bar{P}_n t^n \cdot \sum z_n t^n = \sum (\bar{P} z)_n t^n,$$

so that, if we define $\bar{p}_{-1}=0$, we may write

$$(12)\quad x_n = (\bar{P}z)_n = \sum_{k=0}^{n} (P1)_k \bar{P}_{n-k} y_k = \sum_{k=0}^{n} (P1)_k (\bar{p}_{n-k} - \bar{p}_{n-k-1}) y_k.$$

Thus the matrix $\bar{b}_{nk}$ corresponding to the transformation B^{-1} is given by the equation

$$(13)\qquad \bar{b}_{nk} = (\bar{p}_{n-k} - \bar{p}_{n-k-1})(P1)_k,$$

and the matrix c_{nk} corresponding to $C=AB^{-1}$ is given by the equation

$$c_{nk} = \sum_{h=k}^{n} a_{nh} \bar{b}_{hk},$$

which, on substitution from (7) and (13), gives

$$(14)\qquad c_{nn} = a_{nn} \bar{b}_{nn} = \frac{(P1)_n}{(n+1)P_n},$$

and, for $n>k$,

$$\begin{aligned} c_{nk} &= \frac{(P1)_k}{n+1} \sum_{h=k}^{n} \sum_{i=h}^{n} \frac{p_{i-h}}{P_i} (\bar{p}_{h-k} - \bar{p}_{h-k-1}) \\ &= \frac{(P1)_k}{n+1} \sum_{i=k}^{n} \frac{1}{P_i} \sum_{h=k}^{i} p_{i-h} (\bar{p}_{h-k} - \bar{p}_{h-k-1}). \end{aligned}$$

Now we know that

$$\begin{aligned} \sum_{h=k}^{i} p_{i-h}(\bar{p}_{h-k} - \bar{p}_{h-k-1}) &= \sum_{h=k}^{i} p_{i-h}\bar{p}_{h-k} - \sum_{h=k+1}^{i} p_{i-h}\bar{p}_{h-k-1} \\ &= (p\bar{p})_{i-k} - (p\bar{p})_{i-k-1} = 1, \text{ or } -1, \text{ or } 0, \end{aligned}$$

according as $i = k$, or $i = k+1$, or $i \geqq k+2$. Thus

$$(15)\qquad c_{nk} = \frac{(P1)_n}{n+1}\left(\frac{1}{P_k} - \frac{1}{P_{k+1}}\right) = \frac{(P1)_k p_{k+1}}{(n+1)P_k P_{k+1}}, \qquad (n > k).$$

Since the elements c_{nk} are positive, by (14) and (15), it follows at once that the conditions of the Toeplitz theorem are satisfied, and the matrix $C=AB^{-1}$ is regular. This completes the proof of Theorem 3.

6. *Proof of Theorem* 5. From the transformation $y=C(x)=AB^{-1}(x)$, we have

$$z_n = (n+1)y_n = \sum_{k=0}^{n-1} \frac{(P1)_k p_{k+1}}{P_k P_{k+1}} x_k + \frac{(P1)_n}{P_n} x_n,$$

and

$$z_n - z_{n-1} = \frac{(P1)_n x_n - (P1)_{n-1} x_{n-1}}{P_n};$$

whence

$$x_n = \frac{P_n}{(P1)_n} z_n - \sum_{k=0}^{n-1} \frac{p_{k+1}}{(P1)_n} z_k.$$

Thus the matrix $\bar{c}_{nk}$ corresponding to the transformation $\overline{C}=BA^{-1}$ is given by

$$\bar{c}_{nn} = \frac{(n+1)P_n}{(P1)_n}, \qquad \bar{c}_{nk} = -\frac{(k+1)p_{k+1}}{(P1)_n}, \qquad (n > k).$$

It is easily seen that the condition $\sum_{k=0}^{n}\bar{c}_{nk}=1$ is satisfied. Furthermore,

$$(P1)_n \to \infty,$$

for the series $\sum P_n$ cannot converge, since the general term $P_n > p_0 = 1$. Thus we have for fixed k,

$$\lim_{n\to\infty} \bar{c}_{nk} = 0.$$

Furthermore, we have

$$\sum_{k=0}^{n} |\bar{c}_{nk}| = \bar{c}_{nn} - \sum_{k=0}^{n-1} \bar{c}_{nk} = 2\bar{c}_{nn} - 1.$$

Thus the third Toeplitz condition will be satisfied when and only when condition (6) is satisfied. This proves Theorem 5.

7. *Proof of Theorem* 4. To prove Theorem 4, we shall give an example of a Nörlund transformation for which condition (6) is not satisfied. We define

$$(16)\quad P_0 = 1,\ P_n = \frac{(n)^{1/2}(n-1)^{1/2}\cdots(2)^{1/2}(1)^{1/2}}{[(n+1)^{1/2}-1][(n)^{1/2}-1]\cdots[(2)^{1/2}-1]},$$

$(n>0)$, so that

$$P_n = \frac{(n)^{1/2}}{(n+1)^{1/2}-1} P_{n-1}, \qquad (n > 0).$$

We now see, in turn, that

$$P_n > P_{n-1}, \qquad P_n > 0, \qquad \frac{P_n}{P_{n-1}} \to 1, \qquad \frac{p_n}{P_n} \to 0.$$

Thus the transformation (16) is of the Nörlund type and is regular. We find, however, that

$$(P1)_1 = (2)^{1/2}P_1, \qquad (P1)_2 = (3)^{1/2}P_2,$$

and, by induction,

$$(P1)_n = (n+1)^{1/2}P_n.$$

Thus condition (6) is not satisfied and the proof of Theorem 4 is completed.

The more general question as to the relationship between the transformations QP and $Q \circ P$, where Q and P are *any* two Nörlund transformations, as well as the consideration of the corresponding problem for the transformation of functions instead of sequences, will be treated by the author in a subsequent paper.

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