CONTINUITY IN TOPOLOGICAL GROUPS*

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In the theory of topological groups it is customary to make certain assumptions concerning the continuity of the product and the continuity of the inverse. It will be shown here that for certain types of group spaces less stringent assumptions than those usually made yield the ordinary assumptions as theorems.

Suppose that G is a metric space‡ whose elements form a group. If x and y are any two elements of G, the distance between them will be denoted by $d(x, y)$ and their (group) product will be denoted by $x \cdot y$ or xy. The inverse of x will be denoted by x^{-1}, and the identity of the group by e. If H is a set of elements of G, then xH, Hx, and H^{-1} are sets in G having an obvious definition. The function xy is a function defined everywhere in the product space $G \times G$. It is often assumed that this function is continuous in the two variables simultaneously, but the following theorem shows that in a large class of cases the simultaneous continuity follows from continuity in each variable separately and this with no continuity restriction whatever on the inverse function. In fact it will be shown for separable groups that the continuity of the inverse also follows from the continuity of xy in x and y separately.

THEOREM 1. *If G is locally complete§ and the function xy is continuous in each variable separately, then it is continuous in the two variables simultaneously.*

Let it be noted first that it is sufficient to prove the simultaneous continuity at (e, e),‖ for if there is a discontinuity any-

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‡ Fréchet introduced such spaces. For an account of them see his *Les Espaces Abstraits*. See also Kuratowski, *Topologie*, I.

§ G is said to be locally complete if there is about every point an open set whose closure is complete.

‖ The customary notation for points in a product space is followed here.

where there will be one here. In order to see this let a_n be a sequence approaching a and let b_n be a sequence approaching b, while a_nb_n does not approach ab. Because of the left and right continuity $a^{-1}a_n$ and b_nb^{-1} are sequences approaching e, but $a^{-1}a_nb_nb^{-1}$ does not approach e for if it did we could use first the left and then the right continuity to show that a_nb_n approaches ab.

It is convenient first to prove a lemma.

LEMMA. *If H is any open set in G and ϵ is any positive number, then there exists an open subset H_1 of H and a positive number δ such that for all elements h of H_1 and any element a of G the relation $d(a, e) < \delta$ implies the relation $d(ah, h) \leqq \epsilon$.*

Let B_n denote all elements h of G such that for any a, $d(a, e) < 1/n$ implies $d(ah, h) \leqq \epsilon$. The set B_n is closed, a fact which may be seen as follows. Suppose that B_n is not closed and that b_m is a sequence of elements of B_n approaching an element b not in B_n. Since b is not in B_n, there is some element a such that $d(a, e) < 1/n$ and (1) $d(ab, b) > \epsilon$. For all b_m, however, (2) $d(ab_m, b_m) \leqq \epsilon$. Because xy is continuous in y, $\lim ab_m = ab$. Thus (1) and (2) are contradictory and from this contradiction it may be concluded that B_n is closed.

Because of the left continuity of xy every h in H belongs to B_n for sufficiently large n. Therefore $H \subset \sum_n B_n$. Since H is of the second category,* there must be some n such that $H \cap B_n$† is of the second category. Then B_n must be everywhere dense in some open subset H_1 of H; and from the fact that B_n is closed, B_n must include all of H_1. The lemma is now demonstrated.

The proof of Theorem 1 may now be given. Let G (for uniformity denote G by H_0) be the first open subset of G to which the lemma is applied; by this lemma there exists a positive number δ_1 and an open subset H_1 of H_0 such that for all elements a in G and all elements h in H_1, $d(a, e) < \delta_1$ implies that $d(ah,h) \leqq 1$.

Application of the lemma next to H_1 shows that there is a δ_2 and an open subset H_2 of H_1 such that for all elements a in G and all elements h in H_2, $d(a, e) < \delta_2$ implies $d(ah, h) \leqq 1/2$.

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* Banach, *Théorie des Opérations Linéaires*, p. 14. By hypothesis H contains complete metric subspaces and therefore the statement follows from Banach's theorem at once.

† This denotes the intersection or point set product of H and B_n.

Proceeding in this manner, we obtain for every n a δ_n and an open subset H_n of H_{n-1} such that for all a in G and all h in H_n, $d(a, e) < \delta_n$ implies $d(ah, h) \leqq 1/n$. It may be assumed that the diameter of H_n is less than $1/n$, and that $\overline{H}_{n+1} \subset H_n$, and that for some n, H_n is complete, the last assumption being possible because G is locally complete. Under these conditions, we have $\prod_n \overline{H}_n = \prod_n H_n = h_0$, where h_0 is some point of G.

Let ϵ be any positive number whatever. Since xy is continuous in x and since h_0^{-1} is constant, there is a number δ such that, for any h in G, $d(h, h_0) < \delta$ implies $d(hh_0^{-1}, h_0h_0^{-1}) = d(hh_0^{-1}, e) < \epsilon$. Let n be so large that $1/(2n) < \delta/2$. By the definition of δ_i it is true for all h in H_{2n} that $d(a, e) < \delta_{2n}$ implies $d(ah, h) \leqq 1/(2n)$. Now let $S(e, \delta_{2n})$ be the open sphere of center e and radius δ_{2n} and let $O = [S(e, \delta_{2n})] \cap [H_{2n} \cdot h_0^{-1}]$. The set O is open and includes e. If b is an element of O, $b < hh_0^{-1}$, where h is in H_{2n}. Let a be any other element of O. Then $d(ab, e) = d(ahh_0^{-1}, h_0h_0^{-1})$. But $d(ah, h) \leqq 1/(2n)$, and $d(h, h_0) < 1/(2n)$. Therefore $d(ah, h_0) < 1/n < \delta$, and it follows that

$$d(ab, e) = d(ahh_0^{-1}, h_0h_0^{-1}) = d(ahh_0^{-1}, e) < \epsilon.$$

Hence the function xy is continuous at (e, e), because for an arbitrary ϵ there has been found an open set $O \times O$ including $e \times e$ such that for any element (a, b) in $O \times O$, $d(ab, e) < \epsilon$. By the remark immediately following Theorem 1, it is evident that the proof is now complete.

This theorem could be easily proved if G were assumed to be separable, by making use of known theorems. Since xy is continuous in each variable separately, it is of Baire class 1 in the two variables together. It therefore has points of continuity* and if it has any points of continuity it is continuous everywhere, as can be seen from the remark immediately following the statement of the theorem. In the non-separable case xy is of class 1 as before but whether it has points of continuity does not follow in this case from any known theorem. It would be interesting to know whether or not the next theorem, which is proved for only the separable case, is also true in the non-separable case.

THEOREM 2. *If G is complete and separable and if xy is continuous in x and y separately, then x^{-1} is continuous.*

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* See Kuratowski, loc. cit., pp. 180, 189, for the relevant theorems.

It follows from Theorem 1 that xy is continuous in x and y simultaneously. It will now be shown that x^{-1} is a function in the Baire classification. In order to do this it is sufficient to prove that if F is any closed set in G, then F^{-1} is a Borel set. Let M denote the set of points (x, y) of $G \times G$ such that $xy = e$. This set is closed because of the continuity of xy. Now let $N = (G \times F) \cap M$. The projection of this set on G is F^{-1}. This is because N contains those points of $G \times G$ which are of the form (x, y), where y is in F and $xy = e$. Further, no two points of N project into the same point so that F^{-1} is the continuous (1-1) image of N. Since N is not necessarily compact, it can not be concluded that F^{-1} is closed, but under the present circumstances it can be concluded that F^{-1} is a Borel set* and hence x^{-1} is a Baire function.

The proof of the theorem is now completed by a lemma.

LEMMA. *If G is separable and complete and if xy is continuous in each variable separately and x^{-1} is a Baire function, then x^{-1} is continuous.*

The proof of this lemma follows with little variation the proof of a theorem of Banach.† First note that it is sufficient to prove that x^{-1} is continuous in the neighborhood of e (see Banach). Since x^{-1} is a Baire function it is continuous on a set H, where $G-H$ is of the first category. Let a_n be a sequence of elements in G approaching e. Since $G-H$ is of the first category $a_n^{-1}(G-H)$ is also of the first category. It follows (see Banach) that $G-H \cdot \prod(a_n^{-1}H)$ can not equal G so that there is a point a in H and in $a_n^{-1}H$ for each n. Therefore $a_n a$ is in H. Since $a_n a$ approaches a and since x^{-1} is continuous on H, it follows that $(a_n a)^{-1} = a^{-1}a_n^{-1}$ approaches a^{-1} and that a_n^{-1} approaches e.

Theorem 2 clearly remains true if we replace the hypothesis of completeness by the hypothesis of local completeness.‡

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* Kuratowski, loc. cit., p. 251.

† Banach, loc. cit., p. 23.

‡ In fact, if a space is locally complete, it may be metrized so as to be complete; but for some applications the hypothesis given is more convenient.