

that $m_\nu \neq 0$ for $\nu < k$, while $m_\nu = 0$ for $\nu \geq k$. If $m_1 > 0$, then e_{s_k} is the n th member of the class C . If $m_1 < 0$, then $e_{s_{k-1}}$ is the $(n - m_{k-1})$ st member of C . If $m_1 = 0$, the n th member of C is e_h , where $n \leq h \leq m_0$.

PROOF. If $m_1 = 0$, then equations (2) are all identical and reduce to

$$(3) \quad n = \theta(m_0),$$

all the other m 's are zero. The theorem then follows at once with $k = 1$, the last statement following from (3) which tells us that n members of C have subscripts $\leq m_0$.

From the definition of $\theta(x)$, we have for $x > y$,

$$(4) \quad 0 \leq \theta(x) - \theta(y) \leq x - y.$$

For $m_1 \neq 0$ we consider the difference

$$(5) \quad m_{r-1} - m_r = \theta(s_r) - \theta(s_{r-1}) = \theta(s_{r-1} + m_{r-1}) - \theta(s_{r-1}).$$

Using (4) we see that if $r \geq 2$,

$$(6) \quad m_{r-1} = 0 \text{ implies } m_r = 0,$$

$$(7) \quad m_{r-1} > 0 \text{ implies } 0 \leq m_{r-1} - m_r \leq m_{r-1},$$

$$(8) \quad m_{r-1} < 0 \text{ implies } 0 \geq m_{r-1} - m_r \geq m_{r-1}.$$

From these equations we see that the m 's do not differ in sign. Also (7) shows that the m 's do not increase if positive, while (8) shows that they do not decrease if negative. Therefore the sequence $|m_r|$, ($r = 1, 2, 3, 4, \dots$), is non-increasing. If possible suppose that $|m_r| > 0$, for all $r > 0$; then $s_r = m_0 + m_1 + m_2 + \dots + m_{r-1}$ would either increase or decrease indefinitely according as $m_1 > 0$ or < 0 . In the first case, since C is an infinite class, $\theta(s_r)$ would also increase without limit, so that for r sufficiently large $m_r = n - \theta(s_r)$ would be negative, contrary to fact. In the second case s_r would ultimately become negative so that $\theta(s_r) = 0$. For such an r we would have $m_r = n$. But n is positive, while m_r is not. Hence $|m_r|$ becomes and remains zero for all sufficiently large r , say $r \geq k$.

To complete the proof of the theorem set $r = k$ in (5) and obtain

$$(9) \quad m_{k-1} = \theta(s_{k-1} + m_{k-1}) - \theta(s_{k-1}).$$

If m_1 and hence m_{k-1} is positive, (9) tells us that all the m_{k-1} members of $S: e_{s_{k-1}+1}, \dots, e_{s_{k-1}+m_{k-1}}$ belong to C . In particular the last one which is e_{s_k} belongs to C . Moreover $m_k = n - \theta(s_k) = 0$. Hence e_{s_k} is the n th member of C . If m_1 and hence m_{k-1} is negative, (9) tells us that all the m_{k-1} members of $S: e_{s_{k+1}}, \dots, e_{s_{k-1}}$ belong to C . In particular $e_{s_{k-1}}$ is a member of C . But $m_{k-1} = n - \theta(s_{k-1})$ or $\theta(s_{k-1}) = n - m_{k-1}$; hence $e_{s_{k-1}}$ is the $(n - m_{k-1})$ st member of C . Thus the proof is complete.

If S and C are wholly arbitrary one may show by means of examples that the conclusion of the theorem is the best that can be obtained from the hypothesis. Additions to the hypothesis make the last two statements of the theorem more precise. For example, if we assume that no two consecutive members of C are consecutive members of S , it is easy to prove that $e_{s_{k-1}}$ is the $(n - 1)$ st member of C in case $m_1 < 0$.

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A NEW CLASS OF FUNCTIONS OF TWO VARIABLES INVOLVING BESSEL FUNCTIONS OF HALF AN ODD INTEGER*

BY N. A. HALL

The evaluation of certain integrals arising in the theory of the conduction of heat between two media of different conductivities† suggested the consideration of the expansions:

$$\begin{aligned}
 (1) \quad \sin [\alpha(1 + x^2)^{1/2} - \beta x] &= \sum_{n=0}^{\infty} x^n S_n(\alpha, \beta), \\
 (2) \quad \cos [\alpha(1 + x^2)^{1/2} - \beta x] &= \sum_{n=0}^{\infty} x^n C_n(\alpha, \beta), \\
 (3) \quad \exp i[\alpha(1 + x^2)^{1/2} - \beta x] &= \sum_{n=0}^{\infty} x^n E_n^{(1)}(\alpha, \beta), \\
 (4) \quad \exp -i[\alpha(1 + x^2)^{1/2} - \beta x] &= \sum_{n=0}^{\infty} x^n E_n^{(2)}(\alpha, \beta).
 \end{aligned}$$

* Presented to the Society, November 30, 1935.

† G. Green, *Philosophical Magazine*, (7), vol. 18 (1934), p. 631.