A CERTAIN MEAN-VALUE PROBLEM IN STATISTICS*

BY A. T. CRAIG

1. *Introduction.* It is the purpose of this paper to investigate, by means of the characteristic function, the arithmetic mean value, or mathematical expectation, of the sum of the squares of n normally and independently distributed variables when those variables are subject to $m<n$ linear restrictions. For example, if $x_1, x_2, \cdots, x_n$ are n independent values of a variable x which is normally distributed with mean zero and variance σ^2, then the expected value of $\sum_1^n x_j^2$ is $n\sigma^2$. However, the expected value of $\sum_1^n(x_j-\bar{x})^2$, where $n\bar{x}=\sum_1^n x_j$, is $(n-1)\sigma^2$. It is fairly obvious that the latter example could be stated: if the x's are subject to the linear restriction $\sum_1^n x_j=0$, the expected value of $\sum_1^n x_j^2$ is $(n-1)\sigma^2$. The numbers n and $n-1$, which are equal respectively to the ranks of the matrices of the two quadratic forms, are frequently called the number of degrees of freedom of those quadratic forms.

Let x be subject to the normal law of error

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-x^2/2\sigma^2}$$

and let $x_1, x_2, \cdots, x_n$, be n independent values of x. Write

$$v = \sum_1^n x_j^2, \qquad u_1 = \sum_1^n a_{1j}x_j, \cdots, \qquad u_m = \sum_1^n a_{mj}x_j,$$

in which the a's are real numbers. We wish to find the mathematical expectation of v when $u_1, u_2, \cdots, u_m$ are assigned values which make the system consistent. It is well known that the variables $u_1, u_2, \cdots, u_m$ are normally correlated with variances and covariances given by $\sigma^2\sum_r a_{jr}a_{kr}$.

2. *The Characteristic Function.* The characteristic function of the joint distribution of $v, u_1, \cdots, u_m$ is

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$$\phi(t_1, t_2, \cdots, t_m, t_{m+1}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \int \cdots \int e^{\theta} dx_n \cdots dx_1,$$

where

$$\theta = it_1 \sum_1^n a_{1j}x_j + \cdots + it_m \sum_1^n a_{mj}x_j + \left(it_{m+1} - \frac{1}{2\sigma^2}\right) \sum_1^n x_j^2,$$

and $i = \sqrt{(-1)}$. Throughout this paper we shall understand that the limits of integration are $-\infty$ and ∞ unless otherwise specified. If we write

$$b_{11} = \sum a_{1j}^2, \quad b_{22} = \sum a_{2j}^2, \quad \cdots, \quad b_{mm} = \sum a_{mj}^2,$$
$$b_{12} = b_{21} = \sum a_{1j}a_{2j}, \quad \cdots, \quad b_{m-1,m} = b_{m,m-1} = \sum a_{m-1,j}a_{mj},$$

and

$$Q = \sum_{j,k} b_{jk}t_jt_k,$$

then

$$\phi(t_1, \cdots, t_{m+1}) = \frac{e^{-\sigma^2 Q/2(1-2i\sigma^2 t_{m+1})}}{[1 - 2i\sigma^2 t_{m+1}]^{n/2}}.$$

From this latter result, it is fairly obvious that the problem has no solution unless Q is a positive definite quadratic form of rank m. Upon writing $t_{m+1} = 0$, we find the characteristic function of the joint distribution of the m linear forms to be

$$\phi(t_1, \cdots, t_m, 0) = e^{-\sigma^2 Q/2}.$$

Moreover, if $\psi = \psi(u_1, \cdots, u_m)$ is the simultaneous distribution function of these linear forms, then

$$\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-L-\sigma^2 Q/2} dt_m \cdots dt_1,$$

where

$$L = it_1u_1 + \cdots + it_mu_m.$$

Since Q is positive definite of rank m, the Cayley-Hamilton equation of the matrix $B = \|b_{jk}\|$ of Q has m real positive roots, say $\lambda_1, \lambda_2, \cdots, \lambda_m$. Moreover, there exists a real orthogonal matrix $C = \|c_{jk}\|$ such that

$$C'BC = \begin{Vmatrix} \lambda_1 & 0 & 0 \cdots 0 \\ 0 & \lambda_2 & 0 \cdots 0 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 0 & 0 \cdots 0 & \lambda_m \end{Vmatrix}.$$

If then, in the latter integral, we introduce new variables $z_1, \cdots, z_m$ by subjecting the t's to a linear homogeneous transformation with matrix C, we get

$$\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-iS_1z_1 - \cdots - iS_mz_m - (\sigma^2/2)\Sigma\lambda_j z_j^2} dz_m \cdots dz_1$$

$$= \frac{1}{(\lambda_1 \cdots \lambda_m)^{1/2}(2\pi\sigma^2)^{m/2}} e^{-S_1^2/2\lambda_1\sigma^2 - \cdots - S_m^2/2\lambda_m\sigma^2},$$

where $S_p = \sum c_{pj}u_j$, $(p = 1, 2, \cdots, m)$.

3. *The Mathematical Expectation of v.* Let $F = F(u_1, \cdots, u_m, v)$ be the simultaneous distribution function of v and the m linear forms. Also, let $\bar{v}$ be the expected value of v for $u_1, \cdots, u_m$ assigned. Thus

$$\bar{v} = \int \frac{vF}{\psi} dv,$$

in which the limits of integration on v are here and elsewhere taken to cover all admissible values of that variable when $u_1, \cdots, u_m$ are regarded as assigned. Now

$$\phi(t_1, \cdots, t_m, t_{m+1}) = \int \cdots \int e^{L + ivt_{m+1}} F \, dv \, du_m \cdots du_1,$$

and

$$\left.\frac{\partial \phi}{\partial t_{m+1}}\right|_{t_{m+1}=0} = i \int \cdots \int ve^L F \, dv \, du_m \cdots du_1$$

$$= i \int \cdots \int ve^L \frac{F}{\psi} \psi \, dv \, du_m \cdots du_1$$

$$= i \int \cdots \int \bar{v} e^L \psi \, du_m \cdots du_1.$$

Thus

$$i\bar{v}\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-L} \frac{\partial \phi}{\partial t_{m+1}}\bigg|_{t_{m+1}=0} dt_m \cdots dt_1.$$

But

$$\frac{\partial \phi}{\partial t_{m+1}}\bigg|_{t_{m+1}=0} = i(n\sigma^2 - Q\sigma^4)e^{-\sigma^2 Q/2}.$$

Accordingly,

$$\bar{v}\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int (n\sigma^2 - Q\sigma^4)e^{-L-\sigma^2 Q/2} dt_m \cdots dt_1,$$

$$= n\sigma^2\psi - \sigma^2\psi\left[\left(1 - \frac{S_1^2}{\lambda_1\sigma^2}\right) + \left(1 - \frac{S_2^2}{\lambda_2\sigma^2}\right) + \cdots + \left(1 - \frac{S_m^2}{\lambda_m\sigma^2}\right)\right],$$

and

$$\bar{v} = \sigma^2\left[n - m + \frac{1}{\sigma^2}\left(\frac{S_1^2}{\lambda_1} + \cdots + \frac{S_m^2}{\lambda_m}\right)\right].$$

We now see that if each linear form is set equal to zero, the expected value of v is $\bar{v} = (n-m)\sigma^2$. Thus, when $u_1 = u_2 = \cdots = u_m = 0$, we may say that we lose one degree of freedom for each linear restriction in estimating σ^2 from v.

4. *Independent Linear Restrictions.* Of particular interest is the case in which the variables u_j are not correlated. A necessary and sufficient condition for the independence of the variables u_j is that

$$\phi(t_1, 0, \cdots, 0, 0)\cdot\phi(0, t_2, 0, \cdots, 0) \cdots \phi(0, \cdots, 0, t_m, 0) = \phi(t_1, \cdots, t_m, 0);$$

that is, when $b_{jk} \neq 0, j = k$, and $b_{jk} = 0, j \neq k$. Under these conditions, ψ becomes

$$\psi = \left(\frac{1}{2\pi\sigma^2}\right)^{m/2} \frac{1}{(b_{11} \cdots b_{mm})^{1/2}} e^{-u_1^2/2\sigma^2 b_{11} - \cdots - u_m^2/2\sigma^2 b_{mm}},$$

and

$$\bar{v} = \sigma^2\left[n - m + \frac{1}{\sigma^2}\left(\frac{u_1^2}{b_{11}} + \cdots + \frac{u_m^2}{b_{mm}}\right)\right].$$

Again we observe that the expected value of v is $(n-m)\sigma^2$ when each of the m linear forms is equated to zero. However, if s of the m linear forms are equated to their respective standard derivations while the remaining $m-s$ are equated to zero, then $\bar{v}=(n-m+s)\sigma^2$. Finally we see that the expected value of v, for a fixed set of u's, is not in general an integral multiple of σ^2.

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ON THE PRESERVATION OF ANGLES AT A BOUNDARY POINT IN CONFORMAL MAPPING†

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The object of this note is to prove the following theorem.

THEOREM. *Let R be a simply connected "schlicht" region in the w-plane whose boundary contains the point $w=0$. Let $w=0$ be "accessible" along the Jordan curve L. Suppose that there is a circle $|w|<\rho$ such that the part of the boundary of R which is inside this circle lies within the angles*

$$(1)\qquad |\arg w - h_+| \leqq k_+, \qquad |\arg w - h_-| \leqq k_-, \qquad (h_- \leqq h_+).$$

Suppose, furthermore, that L connects $w=0$ with a boundary point outside $|w|=\rho$ such that L divides R into two sub-regions. Let all boundary points of one sub-region which are in $|w|<\rho$, and not on L, be in one of the angles (1), *and those of the other sub-region which are in $|w|<\rho$, and not on L, be in the other.*

Let $w=w(z)$ map $|z-1|<1$ conformally on R in such a manner that its inverse function approaches 0 *as $w\to 0$ along L. Let*

$$(2)\qquad \begin{aligned} H(\alpha) &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\alpha\right)h_+ + \left(\frac{\pi}{2}-\alpha\right)h_-\right], \\ K(\alpha) &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\alpha\right)k_+ + \left(\frac{\pi}{2}-\alpha\right)k_-\right]. \end{aligned}$$

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