THE VENERONI TRANSFORMATION IN S_n*

BY VIRGIL SNYDER AND EVELYN CARROLL-RUSK

1. *Introduction*. The purpose of the present paper is to obtain the system of bilinear equations of the Veneroni transformation defined by associating projectively the primes (or hyperplanes) of S_n with the primals (or hypersurfaces) V^n of order n through $n+1$ arbitrary S'_{n-2} of S'_n, and to derive a complete scheme of mapping of the manifolds of either space on the linear manifolds of the other. Both of these are believed to be new.

2. *Analytic Expression for the Veneroni Transformation*. The process is somewhat different according as n is odd or even. It will be most easily understood by considering in detail the case $n=5$. In S_5 six three way spaces σ_i, $(i=1, 2, \cdots, 6)$, form the base of a Veneroni transformation. Let $\sigma_1 \equiv x_1=0,\ x_2=0$; $\sigma_2 \equiv x_3=0, x_4=0, \sigma_3 \equiv x_5=0, x_6=0$, and $\sigma_4 \equiv a=\sum_{i=1}^{6} a_i x_i=0, b=0$; $\sigma_5 \equiv c=0,\ d=0$; $\sigma_6 \equiv e=0, f=0$. Through five of the σ_i (not σ_k) passes one and only one V_4^4, say V_k, determined by the ∞^3 lines meeting all of them. This and any S_4 of the pencil through the remaining σ_k provides one composite quintic primal of the system. Thus a complete homaloidal system is obtained. Among these primals six independent identities exist of the form

$$x_2 V_1 = (e_1 f) V_6 + (c_1 d) V_5 + (a_1 b) V_4,$$

in which $(a_i b) = a_i b(x) - b_i a(x), \cdots$. If $\rho x_1' = x_1 V_1$, $\rho x_2' = x_2 V_1$, $\rho x_3' = x_3 V_2, \cdots$, we may write

$$\frac{a(x)}{b(x)} = -\frac{\sum B_i x_i'}{\sum A_i x_i'},$$

and so on where B_i is the cofactor of b_i in the determinant

$$D = \left| a_1 b_2 c_3 d_4 e_5 f_6 \right|,$$

after the transpositions (12), (34), (56) have been made in the subscripts. Any two of these and the three equations

$$x_1 x_2' - x_1' x_2 = 0, x_3 x_4' - x_3' x_4 = 0, x_5 x_6' - x_5' x_6 = 0$$

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completely define the transformation. A similar statement holds for n any odd number. For n, even a slight variation in the procedure must be introduced, since the equations of the σ_i can not all be taken as part of the simplex of reference.

3. *Double Elements of V_{n-1}^{n-1} in S_n.* Any two primals of S_{n-1} of the system, V_α and V_β, have in common all the $[n-2]$-dimensional bases σ_i except σ_α and σ_β; a ruled manifold $R_{n-2}^{(n+1)(n-2)/2}$ of order $(n+1)(n-2)/2$ consisting of the lines intersecting all the σ_i; and a residual manifold $M_{n-2}^{(n-2)(n-3)/2}$ of order $(n-2)(n-3)/2$.

The bases σ_i, σ_k meet in an $[n-4]$-space denoted by σ_{ik}. The spaces $\sigma_{12}, \sigma_{13}, \cdots, \sigma_{1(n+1)}$ are double and lie in σ_1. On an $[n-1]$-primal, such as V_{n+1}, the bases $\sigma_{12}, \sigma_{13}, \cdots, \sigma_{1n}$ lie in an S_{n-2} and are intersected by a ruled manifold $R_{n-4}^{(n-1)(n-4)/2}$. These $[n-4]$-spaces $\sigma_{12}, \sigma_{13}, \cdots, \sigma_{1n}$ and the generators of the ruled manifold are double on V_{n+1} and form the base of a homaloidal system of primals in S_{n-1} in σ_1. The manifolds of this system which are cut by V_α and V_β intersect therefore in a normal variety M_{n-2} of S_{n-2}, which is a double manifold intersecting each base σ_{ik} in $n-3$ points and not intersecting any generator of $R_{n-4}^{(n-1)(n-4)/2}$. In the general case, there are no other double elements. Similarly, the presence of triple and other multiple loci on each V_{n-1}^{n-1} of the system can be obtained.

For $n=5$, on each V_4^4 containing five base three spaces σ_i there are twenty double lines and five double space cubic curves, all lying in the base spaces σ_i. These results were obtained by Eiesland* by the method of differential geometry.

4. *Residual Intersection of Two V_{n-1}^{n-1} in S_n.* The intersection of V_α and V_β, which is of order $(n-1)^2$, consists of $n-1$ of the bases σ_i, of the ruled manifold $R_{n-2}^{(n+1)(n-2)/2}$, and of a residual manifold of order $(n-2)(n-3)/2$.

Each S_{n-1} of the pencil $|\sigma_k|$, $(k\neq\alpha, \beta)$, meets each of the $n-1$ remaining σ_i, $(i\neq\alpha, \beta)$, in an S_{n-3}. These S_{n-3} intersect in pairs in $[n-5]$-spaces S_{n-5}, belonging to a common $[n-3]$-space K_{n-3} which meets each S_{n-3} in an S_{n-4}. This K_{n-3} is also incident to the σ_k and meets σ_α and σ_β. Hence it lies on V_α and V_β and, consequently, on each primal of the pencil. As S_{n-1} describes the pencil $|\sigma_k|$, this K_{n-3} describes the variety $M_{n-2}^{(n-2)(n-3)/2}$. Its equations are

* J. Eiesland, Palermo Rendiconti, vol. 54 (1930), pp. 335–365.

$$\frac{x_1}{x_2} = \frac{(a_3b_2)x_3 + (a_4b_2)x_4}{(a_1b_3)x_3 - (a_4b_1)x_4} = \frac{(a_5b_2)x_5 + (a_6b_2)x_6}{(a_1b_5)x_5 - (a_6b_1)x_6} = \cdots$$
$$= \frac{(a_nb_2)x_n + (a_{n+1}b_2)x_{n+1}}{(a_1b_n)x_n - (a_{n+1}b_1)x_{n+1}}.$$

For $n=5$, if V_5 and V_6 are determined by $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, respectively, the intersection consists of the bases $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, of the three-dimensional variety $R_3{}^9$, and of a residual variety of order three. Each of the S_4 of the pencil $|\sigma_4|$ meets $\sigma_1, \sigma_2, \sigma_3$, in the planes π_1, π_2, π_3, respectively. These planes meet in pairs in the points $\pi_{12}, \pi_{13}, \pi_{23}$ which determine a plane π_{123} meeting each π_i in a line. π_{123} also meets σ_4 in a line and σ_5 and σ_6, each in a point. The plane π_{123} therefore lies on both V_5 and V_6. As the S_4 describes the pencil, the plane π_{123} describes the cubic variety

$$\frac{x_1}{x_2} = \frac{(a_3b_2)x_3 + (a_4b_2)x_4}{(a_1b_3)x_3 - (a_4b_1)x_4} = \frac{(a_5b_2)x_5 + (a_6b_2)x_6}{(a_1b_5)x_5 - (a_6b_1)x_6}.$$

5. *Transformations in S_n Defined by n Bilinear Equations.* The n bilinear equations

$$A_r \equiv \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} a_{ik}^{(r)} x_i x_k' = 0, \quad (r = 1, 2, \cdots, n)$$

define each x_i' as a function of (x), expressed as a determinant of order n, each element being linear. These determinants belong to a matrix $\|a_{ik}^{(r)}\|$ of n rows and $n+1$ columns. They are satisfied by a manifold M of order $(n+1)n/2$ and of dimensionality $n-2$. Thus, for $n=3$, there results a C_6, defining a web of cubic surfaces. Conversely, the system of primals of order n containing M form a homaloidal system.

An S_{n-k}, defined by k primes in (x'), is transformed into a variety of order $n(n-1) \cdots (n-k+1)/k!$ and of dimensionality $n-k$, having in common with M a variety of order $k \cdot {}_nC_{k+1} \cdot (n+1)/(n-k)$ and of dimensionality $n-k-1$.

Thus, if $n=3$, $k=2$, a line of (x') is transformed into a space cubic curve meeting C_6 in eight points. The lines meeting M in n points generate a primal of order n^2-1. No line has $n+1$ points on M apart from the generators.*

* L. Godeaux, Istituto Lombardo Rendiconti, vol. 43 (1910), pp. 116–119.

In addition to the order $(n+1)n/2$ of M the orders of the double, triple, and other multiple manifolds on this given variety M can be expressed by equating to zero all the determinants of order n, $n-1$, $\cdots$, respectively. The hth variety of this system exists if $d-h(d-n+h)\geqq 0$, d being the dimensionality of the space, and this is its dimension. Its order is given by

$$\frac{(d+n-2q)_h(d+n-2q+1)_h\cdots(d+n-q)_h}{(h)_h(h+1)_h\cdots(n)_h},$$

where $n-h=q$;* $(k)_h$ represents the number of combinations of k things, taken h at a time.

6. *Maps of S_1, S_2, $\cdots$ in a General (n, n) of Veneroni Type.*

(a) If the point x describes a straight line, the image x' describes a normal C_n of S_n', as is seen by solving the equations of the line for every x_i in terms of two homogeneous ones, and then regarding these as the parameters in the locus of x'. The curve may also be defined as the locus of intersections of corresponding primes of $n-1$ projective pencils of primes.

(b) If the point x describes a plane, from its equations all the coordinates x_i can be eliminated except three, which appear linearly and homogeneously. The n defining bilinear equations can be interpreted as n projective systems of $|S_{n-1}|$, each having an S_{n-3} for base.

For example, if the coordinates remaining in (x) are replaced by λ, μ, ν, there are n projective systems $\lambda p^{(i)}+\mu q^{(i)}+\nu r^{(i)}=0$, $p^{(i)}=0$, $q^{(i)}=0$, and $r^{(i)}=0$ being the equations of primes in (x') for $i=1, 2, \cdots, n$. Each S_{n-1} defined by one of these equations passes through $p^{(i)}=0$, $q^{(i)}=0$, and $r^{(i)}=0$, hence through an S_{n-3}. By eliminating λ, μ, ν there results a matrix of three rows and n columns, each element being linear in x_i'. The locus common to all these is a surface F_2 of order $n(n-1)/2$. Each x_i' is a rational polynomial of order n in λ, μ, ν. Since a variable double point can not exist in a linear system of plane curves by Bertini's theorem, the intersections of F_2 and each surface S_{n-1} are represented by plane curves of order n with simple base points. Hence F_2 contains $n(n+1)/2$ lines, and the intersection of F_2 and any S_{n-1} is of genus $(n-1)(n-2)/2$.

* C. Segre, Roma, Reale Accademia dei Lincei, Rendiconti, (5), vol. **9**$_2$ (1900), pp. 253–260.

The $n(n+1)/2$ lines on F_2 are the images of the points in which a plane in (x') meets the $n+1$ base $[n-2]$-spaces σ_i' and of those in which it meets the ruled manifold $R'^{(n+1)(n-2)/2}_{n-2}$. The images of the first set of points are the lines meeting n of the base σ_i and the images of the second set are the generators of $R^{(n+1)(n-2)/2}_{n-2}$.

Each surface F_2 of the system meets the ruled manifold $R^{(n+1)(n-2)/2}_{n-2}$ in $(n+1)(n-2)/2$ lines and contains $n+1$ other lines.

Hence the ∞^n sections of the F_2 of the system are mapped upon a plane by the system of curves of order n through $n(n+1)/2$ points.

For an arbitrary plane, the lines on the image surface F_2 are mutually skew. A line in the image (λ, μ, ν) plane is the image of a normal curve C_n of order n in S_n. If the image line passes through a base point, the C_n consists of a normal C_{n-1} of order $n-1$ in S_{n-1} and one of the $n(n+1)/2$ lines intersecting it in one point. The residual section in the S_{n-1} is a curve $C_{n(n-3)/2}$ of order $n(n-3)/2$ having for an image in the (λ, μ, ν) plane a C'_{n-1} through the other $n(n-1)/2$ points; it is of genus $(n-2)(n-3)/2$. Through this $C_{n(n-3)/2}$ there pass ∞^1 primes, each containing a C_{n-1}, hence the $C_{n(n-3)/2}$ lies in an $[n-2]$-dimensional space S_{n-2}, and each S_{n-1} through it meets F_2 in a C_{n-1} which intersects the line not secant to $C_{n(n-3)/2}$ and also the latter curve in $n-1$ points.

Among the lines of the pencil A_i, images of curves C_{n-1} on F_2, there is one through another base point A_k. The curve on the surface F_2 having this line for image consists of two base lines and a curve C_{n-2} of order $n-2$, hence lying in an S_{n-2}. Since the residual plane curve passes through A_k and the composite curve is the image of a prime section of F_2, it follows that the residual curve on F_2 is tangent to the S_{n-3} passing through $n-1$ curves C_{n-2}, and the space of each C_{n-2} is tangent to two residual curves. This residual curve $C_{(n^2-3n+2/4)}$ of order $(n^2-3n+4)/2$ and of genus $(n-3)(n-4)/2$ intersects C_{n-2} in $n-2$ points. There are therefore $n(n-1)/2$ curves C_{n-2} of S_{n-2} lying on F_2, intersecting in pairs in not more than one point; if their image lines meet in the same base points, the associated C_{n-2} do not intersect. The curves $C_{(n^2-3n+4)/2}$ intersect in pairs in $(n-2)(n-3)/2$ points apart from the base points.

Any two F_2 of the system do not have any points in common. The surfaces F_2 do not contain rational curves of order less than $n-2$, apart from the base lines.

For $n=4$, an S_3 meets every pencil of the quartic primals $|\psi_4|$ in a pencil of quartic surfaces, each containing the intersection of the basic two-dimensional surface F_2 of order six and S_3, a curve C_6 of order six and of genus three. Hence each quartic surface is invariant under an infinite discontinuous group of Cremona transformations.*

Similarly, for $n=5$, an S_4 meets a net of quintic primals in a net of quintic surfaces, each passing through the intersection of the basic two-dimensional surface F_2 of order ten (that is, in ten points) and an S_4, a curve C_{10} of order ten and of genus six. An S_4 containing a normal C_4 on F_2 meets it in a residual C_6 of order six and of genus three, which intersects C_4 in four points. But through this residual C_6 pass ∞^1 $|S_4|$, each meeting F_2 in a normal C_4 of S_4; hence the C_6 lies in the base of the pencil $|S_4|$, and the quartic surfaces through it have the same property as that mentioned in the case $n=4$.

This property is true for prime sections of pencils of primals in S_n, since their equations can always be written in determinantal form.

Any S_{n-2} through one of the bases, for example, $p=q=r=0$, meets each homologous S_{n-2} of the other projective systems in $n-1$ spaces S_{n-4}. The pencil of $|S_{n-1}|$ through the homologous S_{n-2} of the base $p=q=r=0$ meets the base in a pencil $|S_{n-3}|$. Similarly for each of the other systems. There are then in the arbitrary S_{n-2} through the base $n-1$ spaces S_{n-4}, axes of projective pencils of $|S_{n-3}|$, each of the form $a+\lambda b=0$. The condition that these $n-1$ spaces are concurrent is expressed by an equation of order $n-1$ in λ. Hence in the given S_{n-2} there are $n-1$ variable points of F_2. Since an S_{n-2} meets F_2 in $(n+1)n/2$ points, it follows that *each base meets F_2 in $(n-1)(n-2)/2$ points in addition to a base curve.*

(c) If (x) describes an S_3, the equations may be written in the form $x_i' = f_i(\lambda, \mu, \nu, \rho)$, each f_i being a polynomial of order n. These three-dimensional varieties are presented by their prime sections, which are expressed in terms of surfaces of order n, all passing through a surface of order $(n-3)\cdot {}_nC_3\cdot(n+1)/3$.

* Snyder and Sharpe, Transactions of this Society, vol. 16 (1915), pp. 62–70.

If the $n+1$ projective systems of lines, planes, $\cdots$, primes with point vertices $A_0, \cdots, A_n$, are considered arbitrarily in S_n, the locus of intersections of corresponding primes of the system is a variety M of order $(n+1)n/2$ and of dimensionality $n-2$. The locus of intersections of corresponding primes of n of the systems, omitting A_0, is a determinantal primal of order n, Φ_0, passing through all the vertices of the system except A_0 and containing M. In the same way primals Φ_i are obtained, $(i=1, 2, \cdots, n)$, each containing M and all the vertices except A_i. The $n+1$ primals are linearly independent.

Similarly, for the loci of intersections of corresponding primes of $n-1, n-2, \cdots, 3, 2$ of the systems.

The synthetic argument made by Todd* for $n=4$ can be extensively generalized to apply for n general. There are n degrees of freedom to account for the ∞^n primes of S_n. There is a base curve of order $(n+1)n/2$, the intersection of a manifold M of order $(n+1)n/2$ and an S_3. The genus of this curve is determined by the postulation. A surface of order n meets a curve of order m and of genus p in $mn-p+1$ points in order to pass through the curve. The postulation of the surface is $[(n+3)(n+2)(n+1)/6]-1-n$, hence

$$\frac{n^2(n+1)}{2} - p + 1 = \frac{(n+3)(n+2)(n+1)}{6} - 1 - n,$$

from which $p=(n-1)(n-2)(2n+3)/6$. *The projective systems of primes through the three spaces S_3' as bases generate a three-dimensional rational variety of S_n mapped on S_3 by means of surfaces of order n passing through a curve of order $(n+1)n/2$ and of genus $(n-1)(n-2)(2n+3)/6$.* These varieties have $(n+1)\ n(n-1)/6$ singular points, mapped by the multisecants of the base curve.

7. *The Base Elements of the Veneroni Transformation.* The $n+1$ base S_{n-2} form part of the base $[n-2]$-dimensional manifold M of order $(n+1)n/2$, defined by the matrix $\|a_{ik}\|$; the residual is a ruled variety R of order $(n+1)(n-2)/2$ formed by the lines which meet all the $n+1$ base S_{n-2}.

* J. A. Todd, Proceedings of the Cambridge Philosophical Society, vol. 26 (1930), pp. 323–333.

Each normal C_n image of a line meets each base S_{n-2} in $n-2$ points and does not intersect the ruled variety.

The images of planes intersect R in $(n+1)(n-2)/2$ lines. The plane meets each base S_{n-2} in a point, the image of which is a line meeting n of the base S'_{n-2} and lying on F_2. Each base S_{n-2} meets R in a manifold of dimensionality $n-3$ and of order $n-1$. For $n=4$, the two-dimensional variety of order 5 has an infinite number of plane elliptic cubic curves, but the corresponding property is not true for larger values of n although the intersections of each base S_{n-2} and R are birationally equivalent.

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ON THE CHARACTERISTIC ROOTS OF MATRIC POLYNOMIALS*

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1. *Introduction.* Unless otherwise stated, all matrices and polynomials are assumed to have coefficients in an arbitrary algebraically closed field K.

Let A and B denote square matrices of order n. If the characteristic roots of every polynomial $f(A, B)$ are all of the form $f(\lambda, \mu)$, where λ and μ are characteristic roots of A and B, respectively, then in accordance with a notation to be introduced below, we shall say that the matrices A, B have property I_n. By a theorem of Frobenius,† the matrices A, B have this property if they are commutative, but this is by no means a necessary condition. The study of pairs of matrices having property I_n has been the subject of papers by Bruton, Ingraham, and Roth.‡ However, in no case have conditions been obtained which are both necessary and sufficient for the existence of this property.

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† G. Frobenius, *Über vertauschbare Matrizen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin, 1896, pp. 601–614.

‡ The papers by Bruton and by Ingraham have not yet been published in full but abstracts are available as follows: G. S. Bruton, *Certain aspects of the theory of equations for a pair of matrices*, this Bulletin, vol. 38 (1932), p. 633; M. H. Ingraham, *A study of certain related pairs of square matrices*, this Bulletin, vol. 38 (1932), pp. 633–634. Roth's paper is *On the characteristic values of the matrix f(A, B)*, Transactions of this Society, vol. 39 (1936), pp. 234–243.