The first four peaks give the greatest integers requiring 548, 333, 314, 309 ninth powers, respectively. There is very strong evidence that a like result holds for the next 15 peaks. For example, all integers between $e+d$ and $e+2d$ are sums of 128 ninth powers; all between $e+2d$ and $e+3d$ are sums of 125.

UNIVERSITY OF CHICAGO

MAPS OF ABSTRACT TOPOLOGICAL SPACES IN BANACH SPACES*

BY A. D. MICHAL AND E. W. PAXSON

1. *Introduction.* This paper is to serve as a brief introduction to the method of considering the analysis of abstract topological spaces through the medium of homeomorphic mappings of these spaces on subsets of Banach spaces.† Our primary objective here, however, is to obtain for some general topological groups the abstract correspondents of the fundamental Lie partial differential equations for an r-parameter continuous group.‡ The essential notion is the treatment of the general situation with the aid of abstract *coordinates* in Banach spaces wherein the Fréchet differential may be used.§

By an abstract topological space is meant here a set of elements of completely unspecified nature, together with an undefined concept, that of neighborhood of an element (we denote the elements by small Latin letters, and the neighborhood associated with an element a by $U(a)$), satisfying the four Hausdorff postulates given below.||

---

* Presented to the Society, November 30, 1935.

† S. Banach, *Théorie des Opérations Linéaires*, 1932.

‡ S. Lie, *Theorie der Transformationsgruppen*, vols. 1, 3.

§ M. Fréchet, *Annales de l'École Normale Supérieure*, (3), vol. 42 (1925), p. 293. Briefly, $f(x)$ on B_1 to B_2 has a differential at $x=x_0$, if there exists a function $f(x; z)$ on B_1^2 to B_2, linear (additive and continuous) in z and such that given a $\rho>0$ there is determined a $n>0$, so that $\|f(x_0+z)-f(x_0)-f(x_0; z)\| \leq \rho\|z\|$ for $\|z\| \leq n(\rho)$; $f(x_0; z)$ is the differential. See also various papers by Hildebrandt, Graves, Kerner, Michal, and many others.

|| F. Hausdorff, *Mengenlehre*, 1927, pp. 226–229.

(1) $a \epsilon U(a).$

(2) $b \epsilon U(a). \supset : \exists U(b). U(b) \subset U(a).$

(3) $U_1(a) \cap U_2(a) \supset U_3(a).$

(4) $a \neq b. \supset :. \exists U(a). U(b): U(a) \cap U(b) = (0).$

A Banach space is a linear, normed abstract space with real multipliers. If the multipliers are complex, one speaks of a complex Banach space. The fundamental notions of point-set theory and the logical calculus of classes is assumed. By a homeomorphic mapping we mean the customary bi-unique, bi-continuous correspondence of one set of elements to another.*

2. *Types of Mapping.* The first type of mapping between the two spaces is effected as follows. Let Σ be an open subset in the Banach space B. Map homeomorphically every neighborhood in the topological space T on this same set Σ. Note that such a mapping implies the postulation of an infinite character for the space T, since, using postulate (2) above, one has $U(a)h\Sigma$, and $U(b)h\Sigma$ with $U(b) \subset U(a)$. But since homeomorphism is transitive, $U(a)hU(b)$. Then $U(a)$ has the same potency as one of its proper subsets, so that, in the Dedekind sense, it is infinite.

Now consider the intersection C of two neighborhoods $U(\bar{a})$, $U(\bar{b})$, $C = U(\bar{a}) \cap U(\bar{b})$. Then by the mappings† $\alpha = f_{\bar{a}}(c)$, $\beta = f_{\bar{b}}(c)$ one has $\alpha = f_{\bar{a}}(f_{\bar{b}}^{-1}(\beta)) = \phi(\beta)$. Thus $\phi(\beta)$ is a homeomorphic map of Σ^1 on Σ^2, since its composing functions are homeomorphisms. Hence we have the following theorem.

THEOREM 1. *The intersection of two neighborhoods in T determines a homeomorphic mapping of one subset Σ^1 of Σ on to another, Σ^2.*

We now make the following definitions:

DEFINITION 1. The value α of $f_{\bar{a}}(a)$ will be called the abstract coordinate of a in the Banach space B.

DEFINITION 2. The class of homeomorphic transformations $\{\alpha = \phi(\beta)\}$ will be called the class of abstract coordinate transformations for the class $\{C\}$.

---

* W. Sierpinski, *General Topology*, 1934. When two sets E_1, E_2 are homeomorphic under a mapping function f, one writes $E_1 h_f E_2$.

† If $x \epsilon U(\bar{x})$, we write $\alpha = f_{\bar{x}}(x)$, $x = f_{\bar{x}}^{-1}(\alpha)$; $x \epsilon T$, $\alpha \epsilon \Sigma$.

All possible intersections of neighborhoods in T determine, by Theorem 1, homeomorphic mappings of various parts of Σ on various other parts.

DEFINITION 3. If all such point transformations possess Fréchet differentials up to and including the rth $(r \geqq 1)$, the complete mapping will be called r-differentiable.

THEOREM 2. *For a complex Banach space a* 1-*differentiable map is analytic.*

PROOF. By known theorems,* the existence of a first Fréchet differential implies that the function is continuous, that it possesses a Gateaux differential, and hence that the function, and so the map, is analytic.

THEOREM 3. *Let* $c = t(d)$ *map homeomorphically* C^1 *on* C^2, *where* C^1, $C^2 \subset C$. *Then there are induced two distinct homeomorphic maps of two distinct subsets of* Σ *on two other distinct subsets.*

PROOF. $\alpha = f_{\bar{a}}(c)$, $\beta = f_{\bar{b}}(c)$; $\gamma = f_{\bar{a}}(d)$, $\delta = f_{\bar{b}}(d)$. Then $\alpha = f_{\bar{a}}(t(f_{\bar{a}}^{-1}(\gamma))) = \theta_{\bar{a}}(\gamma)$ homeomorphically, and $\beta = f_{\bar{b}}(t(f_{\bar{b}}^{-1}(\delta)))$ $= \theta_{\bar{b}}(\delta)$ homeomorphically, where α, β, γ, δ are typical elements in distinct subsets of Σ.

THEOREM 4. *Every homeomorphic point transformation on* Σ^1 *to* Σ^2 $(\Sigma^1, \Sigma^2 \subset \Sigma)$ *is an abstract coordinate transformation.*

PROOF. Let the transformation be $\delta = \psi(\mu)$. Then there exists a neighborhood $U(\bar{x})$ such that for $z \in U(\bar{x})$, $\mu = f_x(z)$. Hence $\delta = \psi(f_{\bar{x}}(z))$, or there exists a mapping function distinct from $f_{\bar{x}}$, so that $\delta = f_{\bar{y}}(z)$. This implies that there exists a $U(\bar{y}) h_{f_{\bar{y}}} \Sigma$, where $z \in U(\bar{x}) \cap U(\bar{y})$. Hence $\delta = \psi(\mu)$ is a true coordinate transformation.

For an r-differentiable map, considering Theorems 3 and 4, one may make the following definitions.

DEFINITION 4. A homeomorphic point transformation $c = t(d)$ on C^1 to C^2 in the intersection C will be said to be of class $C^{(r)}$.

DEFINITION 5. For a complex Banach space and a 1-differentiable map, $c = t(d)$ will be called an analytic transformation. (See Theorem 2.)

---

* A. E. Taylor, California Institute of Technology thesis, (1936); see also this Bulletin, vol. 41 (1935), p. 800, abstract 41-11-424, and Graves, this Bulletin, vol. 41 (1935), pp. 641-662.

As an example of these definitions take the conformal mapping of one complex plane (T) on another (B). The interiors of circles about points are taken as neighborhoods in T, each one going over into the same suitably chosen open set in B. Then as is readily seen, for analytic mapping, $c=t(d)$ is analytic, in the ordinary sense.

THEOREM 5. *Every $c=t(d)$ (as in Definition 4) determines four points, distinct from c and d, each one of which is in a neighborhood intersection in T.*

PROOF. $\alpha=f_a(c)$, $\beta=f_b(c)$; $\gamma=f_a(d)$, $\delta=f_b(d)$. Hence $\alpha=f_a(t(f_{\bar{a}}^{-1}(\gamma)))=\theta_a(\gamma)$ and $\beta=\theta_b(\delta)$. Also, by the transitivity of homeomorphism $\alpha=\phi_a(\delta)$, $\beta=\phi_b(\gamma)$. By Theorem 4, these represent coordinate transformations for four distinct points in T.

DEFINITION 6. A regular abstract coordinate transformation of class r is an abstract coordinate transformation possessing differentials up to and including the rth, where the rth differential is continuous.

3. *Group Function.* Now we let the topological space T support a group under the composition function $c=g(a,b)$. Further let this group be right-continuous. That is, if a be fixed, and c', b' are variables so that $c'=g(a,b')$, the existence of a $U(c)$ implies the existence of a $U(b)$ such that $U(c)\supset g(a, U(b))$. Hence, as is known, $g(a, U(b))$ is a neighborhood of c, say $U(c)$. Now take any three elements of T, a, b, c and two fixed elements f, h. Then the functions $c'=g(f,b')$, $b'=g(h,a')$ may be considered. One has $c'=g(f,g(h,a'))=g(g(f,h),a')=g(k,a')$.

If a' ranges over at least part of some $U(\bar{a})$, it follows that there will be a $U(\bar{b})$ over part of which b' will range, and a $U(\bar{c})$ over part of which c' will range, by continuity. Then there exist mapping functions $f_{\bar{a}}$, $f_{\bar{b}}$, $f_{\bar{c}}$ such that $\alpha'=f_{\bar{a}}(a')$, $\beta'=f_{\bar{b}}(b')$, $\gamma'=f_{\bar{c}}(c')$, wherein α', β', γ' range over at least parts of Σ. The group function g defines a homeomorphism of c to b in $c=g(a,b)$. Then

$$c' = g(f, b'):\gamma' = f_{\bar{c}}(g(f, f_{\bar{b}}^{-1}(\beta'))) = \mu(\beta'),$$
$$b' = g(h, a'):\beta' = f_{\bar{b}}(g(h, f_{\bar{a}}^{-1}(\alpha'))) = \nu(\alpha'),$$
$$c' = g(k, a'):\gamma' = f_{\bar{c}}(g(k, f_{\bar{a}}^{-1}(\alpha'))) = \rho(\alpha').$$

where $\gamma' h_\mu \beta'$, $\beta' h_\nu \alpha'$, and hence $\gamma' h_{\mu\nu}\alpha'$, that is, $\gamma' = \mu(\nu(\alpha'))$ $= \rho(\alpha')$. But by Theorem 4, there are determined e_1', e_2', e_3'; $e_1' \epsilon U_1 \cap \bar{U}_1, \cdots$, where the e's are not, in general, in any of the neighborhoods above, and for which μ, ν, ρ are coordinate transformations

$$\gamma' = f_1(e_1'), \quad \beta' = \bar{f}_1(e_1'), \cdots$$

Consequently, if the whole mapping is r-differentiable, one has

$$\text{(A)} \qquad \rho(\alpha'; \delta\alpha') = \mu(\nu(\alpha'); \nu(\alpha'; \delta\alpha')).$$

Write

$$\text{(A}') \qquad \begin{aligned} \rho(\alpha'; \delta\alpha') &= R(\gamma', \alpha', \delta\alpha'), \\ \gamma' &= f_{\bar{c}}(c'), \quad \alpha' = f_{\bar{a}}(a'). \end{aligned}$$

Consider a neighborhood $U(\bar{c}_1)$ that intersects $U(\bar{c})$, and a $U(\bar{a}_1)$ that intersects $U(\bar{a})$. Then there are determined for certain subsets of values of c', a' coordinate transformations. In fact

$$\gamma' = f_{\bar{c}}(c'), \quad \gamma_1' = f_{\bar{c}_1}(c'), \qquad (\gamma_1' \neq \gamma'),$$

for values of c' satisfying both of these. Similarly

$$\alpha' = f_{\bar{a}}(a'), \quad \alpha_1' = f_{\bar{a}_1}(a'), \qquad (\alpha_1' \neq \alpha').$$

Then

$$\gamma_1' = f_{\bar{c}_1}(f_{\bar{c}}^{-1}(\gamma')) = \theta(\gamma')$$

and

$$\alpha_1' = f_{\bar{a}_1}(f_{\bar{a}}^{-1}(\alpha')) = \tau(\alpha')$$

are the coordinate transformations. Note that the ranges of possible values for γ' and α' are not as extensive as before.

But between any γ_1' and any α_1' there exists a coordinate transformation determining a point p in the intersection of two $U(\bar{x})$, $U(\bar{y})$, with $\gamma_1' = f_x(p')$, $\alpha_1' = f_{\bar{y}}(p')$, $\gamma_1' = f_{\bar{x}}(f_{\bar{y}}^{-1}(\alpha_1')) = \phi(\alpha_1')$, wherein ϕ has a first Fréchet differential. The coordinate transformation mentioned above is necessarily the same for the whole set γ_1' into the whole set α_1'. For, writing $\phi(\alpha_1')$ explicitly, we have

$$\gamma' = f_{\bar{c}}(g(k, f_{\bar{e}}^{-1}(\alpha'))) = \rho(\alpha') \text{ so that } \gamma_1' = \theta(\rho(\tau^{-1}(\alpha_1'))),$$

that is,

$$\gamma_1' = f_{\bar{c}_1}(f_{\bar{c}}^{-1}(f_{\bar{c}}(g(k, f_{\bar{a}}^{-1}(f_{\bar{a}}(f_{\bar{a}_1}^{-1}(\alpha_1'))))))),$$
$$\gamma_1' = f_{\bar{c}_1}(g(k, f_{\bar{a}_1}^{-1}(\alpha_1'))) = \rho_1(\alpha') = \phi(\alpha_1').$$

(Note the formal invariance of the ρ function.) Hence we have

$$\text{(B)} \qquad \rho_1(\alpha_1'; \delta\alpha_1') = \theta(\rho(\tau^{-1}(\alpha_1')); \rho(\tau^{-1}(\alpha_1'); \tau^{-1}(\alpha_1'; \delta\alpha_1'))).$$

Now (A) may be written

$$\text{(C)} \qquad R_1(\gamma_1', \alpha_1', \delta\alpha_1') = \theta(\gamma'; R(\gamma', \alpha', \delta\alpha')),$$

since $\tau^{-1}(\alpha_1'; \delta\alpha_1') = \delta\alpha'$, where $\gamma_1' = \theta(\gamma')$.

Equation (C) indicates that $R(\gamma', \alpha', \delta\alpha')$ transforms like an abstract contravariant tensor of rank 1, with respect to γ'. With respect to α' and $\delta\alpha'$ the function is scalar.

If in (A′) one puts $\alpha' = \gamma'$,

$$\rho(\alpha'; \delta\alpha') = R(\alpha', \alpha', \delta\alpha') = \delta\alpha'.$$

Further, from (A) and (A′),

$$R(\gamma', \alpha', \delta\alpha') = R(\gamma', \beta', R(\beta', \alpha', \delta\alpha')).$$

Hence, if $\gamma' = \alpha'$, $R(\gamma', \beta', \delta\beta')$ is inverse to $R(\beta', \gamma', \delta\beta')$. So $R(\gamma', \beta', \delta\alpha')$ is a solvable linear function of $\delta\alpha'$ with inverse $R(\beta', \gamma', \delta\alpha')$. R vanishes if and only if $\delta\alpha' = 0$. We remark that for an r-differentiable map the quantities R possess continuous Fréchet differentials up to the $(r-2)$th order.

Now $\mu(\beta'; \delta\beta') = R(\gamma', \alpha', R(\alpha', \beta', \delta\beta'))$, where $\gamma' = \mu(\beta')$, so that μ is independent of α'. Then we may put $\alpha' = \alpha_0$, where α_0 is the correspondent in Σ of the identity element a_0 of the topological group. Define

$$R(\gamma', \alpha_0, \delta\beta') .=. R'(\gamma', \delta\beta'),$$
$$R(\alpha_0, \gamma', \delta\beta') .=. R(\gamma', \delta\beta'),$$

where the R and R' on the right are inverse. *Then the fundamental differential equation of Lie type, satisfied representatively in the Banach space by the continuous topological group function, is*

$$\text{(D)} \qquad \mu(\beta'; \delta\beta') = R'(\mu(\beta'), R(\beta', \delta\beta')).$$

California Institute of Technology