A REMARK ON THE ODD SCHLICHT FUNCTIONS

BY M. S. ROBERTSON

Let (S) denote the class of analytic functions

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \tag{1}$$

regular and univalent or schlicht for $|z| < 1$, and (U) the subclass of odd schlicht functions

$$\phi(z) = [f(z^2)]^{1/2} = z + b_3z^3 + b_5z^5 + \cdots . \tag{2}$$

If $\phi(z)$ is real on the real axis, it has been shown* by J. Dieudonné that for all n

$$|b_{2n-1}| + |b_{2n+1}| \leqq 2, \qquad |b_3| \leqq 1. \tag{3}$$

This is not known to be true in the case where the coefficients are complex except for $n=1$. For complex coefficients it is known† that

$$|b_3| \leqq 1, \qquad |b_5| \leqq e^{-2/3} + \frac{1}{2} \qquad (> 1), \tag{4}$$

from which we could conclude only that

$$|b_3| + |b_5| \leqq \frac{3}{2} + e^{-2/3} \qquad (> 2).$$

It is the purpose of this paper to establish the inequality (3) for $n=2$ for the case when the coefficients are complex numbers; and to show further that

$$\frac{|b_3| + |b_5|}{2} \leqq \left(\frac{|b_3|^2 + |b_5|^2}{2}\right)^{1/2} \leqq 1, \tag{5}$$

$$|a_3| \leqq 1 + |b_3|^2 + |b_5|^2 \leqq 3. \tag{6}$$

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* See J. Dieudonné, Annales de l'Ecole Normale Supérieure, vol. 48 (1931), p. 318.

† See M. Fekete and G. Szegö, Journal of the London Mathematical Society, vol. 8 (1933), pp. 85–89.

The inequality $|a_3| \leqq 3$ is well known,* but the second half of the inequality (6) is new, as far as the author is aware.

Since

$$a_n = \sum_{k=1}^{n} b_{2k-1}b_{2(n-k)+1}, \qquad b_1 = 1,$$

we have by Schwarz's inequality

$$(7) \qquad |a_n| \leqq \sum_{k=1}^{n} |b_{2k-1}|^2,$$

and in particular,

$$|a_3| \leqq 1 + |b_3|^2 + |b_5|^2.$$

It is known† that there exists an absolute constant A such that $|b_{2n+1}| \leqq A$ for all n. The conjecture of Paley and Littlewood that $A = 1$ was found to be false by the example of Fekete and Szegö, who demonstrated the existence of an odd function univalent for $|z| < 1$ for which $|b_5| = e^{-2/3} + 1/2 > 1$. We wish to point out that a weaker statement of the conjecture that $|b_{2n+1}| \leqq 1$ might conceivably be true, namely, that

$$(8) \qquad \sum_{k=1}^{n} |b_{2k-1}|^2 \leqq n, \qquad b_1 = 1.$$

If this weaker conjecture is correct, then by (7) the well known conjecture of L. Bieberbach, $|a_n| \leqq n$, would also be true. To substantiate the weaker conjecture (8), we shall demonstrate here that (8) is true for $n = 3$. It is already known to be true for $n = 1, 2$ by (4).

The method used is that of Fekete and Szegö,‡ who employ the representation of the coefficients of (1) obtained by K. Löwner.§ Denoting a continuous function of absolute value unity by $k(t)$, we have the following representation which Löwner obtained for the coefficients:

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* See K. Löwner, Mathematische Annalen, vol. 89 (1923), pp. 103–121.

† See R. Paley and J. Littlewood, Journal of the London Mathematical Society, vol. 7 (1932), pp. 167–169.

‡ See M. Fekete and G. Szegö, loc. cit.

§ See K. Löwner, loc. cit.

$$a_2 = -2\int_0^\infty k(t)e^{-t}dt,$$

$$a_3 = 4\left[\int_0^\infty k(t)e^{-t}dt\right]^2 - 2\int_0^\infty k^2(t)e^{-2t}dt,$$

$$b_3 = \frac{a_2}{2} = -\int_0^\infty k(t)e^{-t}dt,$$

$$b_5 = \frac{a_3}{2} - \frac{a_2^2}{8} = \frac{3}{2}\left[\int_0^\infty k(t)e^{-t}dt\right]^2 - \int_0^\infty k^2(t)e^{-2t}dt.$$

Let

$$b_5 = |b_5|\, e^{2i\beta} \ (\beta \text{ real}), \quad b_3 = |b_3|\, e^{i(\alpha+\beta)} \ (\alpha \text{ real}), \quad k(t)e^{-i\beta} = e^{i\theta(t)},$$

where $\theta(t)$ is real and continuous save for a finite number of points. Then

$$\begin{aligned} |b_5| &= \mathfrak{R}\left\{\left[\frac{3}{2}\int_0^\infty e^{-t}\cdot e^{i\theta(t)}dt\right]^2 - \int_0^\infty e^{-2t}\cdot e^{2i\theta(t)}dt\right\} \\ &= \frac{3}{2}\left[\int_0^\infty e^{-t}\cos\theta(t)dt\right]^2 - \frac{3}{2}\left[\int_0^\infty e^{-t}\sin\theta(t)dt\right]^2 \\ &\quad - 2\int_0^\infty e^{-2t}\cos^2\theta(t)dt + \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} |b_3|^2 &= \left[\int_0^\infty e^{-t}\cos\{\theta(t) - \alpha\}dt\right]^2 \\ &\quad + \left[\int_0^\infty e^{-t}\sin\{\theta(t) - \alpha\}dt\right]^2. \end{aligned}$$

Since the left-hand side of this equation is independent of α, we have

$$|b_3|^2 = \left[\int_0^\infty e^{-t}\cos\theta(t)dt\right]^2 + \left[\int_0^\infty e^{-t}\sin\theta(t)dt\right]^2.$$

Let x denote the non-negative real root of the equation

$$\left(x + \frac{1}{2}\right)e^{-2x} = \int_0^\infty e^{-2t}\cos^2\theta(t)dt.$$

Then, by the theorem of Valiron-Landau,* we have

$$\left| \int_0^\infty e^{-t} \cos \theta(t) dt \right| \leqq (x+1)e^{-x}.$$

Let

$$A = A(x) \equiv \left| \int_0^\infty e^{-t} \sin \theta(t) dt \right| \leqq 1.$$

It follows that $|b_3|^2 + |b_5|^2 \leqq P(x)$, where

$$\begin{aligned} P(x) &\equiv \left[\frac{3}{2}(x+1)^2 e^{-2x} - \frac{3}{2}A^2 - 2\left(x + \frac{1}{2}\right)e^{-2x} + \frac{1}{2} \right]^2 \\ &\quad + \left[(x+1)^2 e^{-2x} + A^2 \right] \\ &= \left[(3x^2 + 2x + 1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2} \right]^2 \\ &\quad + (x+1)^2 e^{-2x} + A^2. \end{aligned}$$

CASE 1. Suppose

$$0 \leqq A^2 \leqq \frac{2}{3}\left[(3x^2 + 2x + 1)e^{-2x} + \frac{1}{3} \right].$$

Then

$$P(x) \leqq \left[(3x^2 + 2x + 1)\frac{e^{-2x}}{2} + \frac{1}{2} \right]^2 + (x+2)^2 \cdot e^{-2x}.$$

The maximum of the right-hand side of this inequality is 2 and occurs for $x = 0$. Hence in this case

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 2.$$

CASE 2. Suppose

$$\begin{aligned} \frac{2}{3}\left[(3x^2 + 2x + 1)e^{-2x} + \frac{1}{3} \right] &\leqq A^2 \\ &\leqq \frac{1}{3}\left[(3x^2 + 2x + 1)e^{-2x} + 1 \right]. \end{aligned}$$

* See E. Landau, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634, especially pp. 630–632.

Since $|b_3|^2 \leqq 1$, we have, within the range of A^2 in this case,

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2}\right]^2$$

$$\leqq 1 + \left[\frac{1 - 3(3x^2+2x+1)e^{-2x}}{6}\right]^2$$

$$< 2, \qquad \text{(for all } x \geqq 0).$$

CASE 3. Suppose

$$\frac{1}{3}\left[(3x^2+2x+1)e^{-2x}+1\right] \leqq A^2 \leqq 1.$$

Then

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2}\right]^2$$

$$\leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3}{2}\right]^2$$

$$\leqq 2, \qquad \text{(for all } x \geqq 0).$$

Since these cases exhaust all those possible, we have

$$|b_3|^2 + |b_5|^2 \leqq 2,$$

and the equality sign occurs for the function $z/(1-e^{i\alpha}z^2)$.

An application of Schwarz's inequality gives also

$$|b_3| + |b_5| \leqq 2\left(\frac{|b_3|^2 + |b_5|^2}{2}\right)^{1/2} \leqq 2.$$

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