CONGRUENCES WITH A COMMON MIDDLE ENVELOPE*

BY MALCOLM FOSTER

1. *Introduction.* Let C and $\bar{C}$ be two rectilinear congruences whose corresponding rays l and $\bar{l}$ are parallel; and let M be the point on the unit sphere S at which the normal is parallel to l and $\bar{l}$. We refer the sphere to any isothermal system and take the linear element in the form $ds^2=e^{2\lambda}(du^2+dv^2)$.† Relative to the moving trihedral at M, whose x axis is chosen tangent to the curve $v=$const., the coordinates of the points in which l and $\bar{l}$ pierce the xy plane will be denoted by (a, b) and $(\bar{a}, \bar{b})$, respectively. Distances on l and $\bar{l}$ will be measured from these points, and the positive direction will be that which corresponds to the outward-drawn normal at M.

It is the purpose of this note to consider such pairs of congruences as C and $\bar{C}$ when they have a common middle envelope, that is, when the distances to the middle points on l and $\bar{l}$ are equal.

2. *Condition that C and $\bar{C}$ have a Common Middle Envelope.* A necessary and sufficient condition that C and $\bar{C}$ have a common middle envelope is that‡

$$\frac{\dfrac{\partial a}{\partial u}+\dfrac{\partial b}{\partial v}+ar_1-br+2\xi}{p_1}=\frac{\dfrac{\partial \bar{a}}{\partial u}+\dfrac{\partial \bar{b}}{\partial v}+\bar{a}r_1-\bar{b}r+2\xi}{p_1}.$$

This may be written

$$\frac{\partial}{\partial u}(a-\bar{a})+\frac{\partial}{\partial v}(b-\bar{b})+(a-\bar{a})\frac{\partial\lambda}{\partial u}+(b-\bar{b})\frac{\partial\lambda}{\partial v}=0,$$

which, upon multiplication by e^{λ}, becomes

$$\frac{\partial}{\partial u}\left[e^{\lambda}(a-\bar{a})\right]=-\frac{\partial}{\partial v}\left[e^{\lambda}(b-\bar{b})\right];$$

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† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ Foster, loc. cit., p. 163, equation (17).

hence

$$(1) \qquad a - \bar{a} = e^{-\lambda} \frac{\partial R}{\partial v}, \qquad b - \bar{b} = - e^{-\lambda} \frac{\partial R}{\partial u},$$

where R is an arbitrary function of u and v. From (1) we have the following theorem.*

THEOREM 1. *A necessary and sufficient condition that the congruences C and $\bar{C}$ have a common middle envelope is that the congruence defined by the point $(a-\bar{a}, b-\bar{b})$ has for its middle envelope a point, namely, the center of S.*

3. *Rotated Congruences.* Let C be the congruence defined by (a, b); and let this point be rotated through an angle $\pi/2$ about the corresponding normal to the point $(-b, a)$.† If $\bar{C}$ be the congruence defined by the point $(-b, a)$, we say C and $\bar{C}$ constitute a pair of rotated congruences. We wish to determine those congruences $C(a, b)$, which with $\bar{C}(-b, a)$, have a common middle envelope. From (1) we must have

$$a + b = e^{-\lambda} \frac{\partial R}{\partial v}, \qquad b - a = - e^{-\lambda} \frac{\partial R}{\partial u}.$$

The solution of these simultaneous equations will obviously give us the required condition:

$$(2) \qquad a = \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v} + \frac{\partial R}{\partial u}\right), \qquad b = \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v} - \frac{\partial R}{\partial u}\right).$$

We therefore have the following result.

THEOREM 2. *A necessary and sufficient condition that a congruence $C(a, b)$ and its rotated congruence $\bar{C}$ have a common middle envelope is that a and b have the values given in* (2).

Suppose now that $C(a, b)$ has for its middle envelope the center of S. Then $a = e^{-\lambda}(\partial R/\partial v)$, $b = -e^{-\lambda}(\partial R/\partial u)$. If $C(a, b)$ be rotated to $\bar{C}(-b, a)$, we know that $\bar{C}$ is a normal congruence.‡

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* Foster, loc. cit., p. 173.

† The direction of rotation is immaterial.

‡ Foster, loc. cit., p. 166, Theorem 1.

Let us now consider the middle point of the line joining (a, b) and $(-b, a)$; its coordinates are $[(a-b)/2, (a+b)/2]$, or

$$\left[\frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v}+\frac{\partial R}{\partial u}\right), \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v}-\frac{\partial R}{\partial u}\right)\right]. \tag{3}$$

Since (3) is identical with (2), we have the following theorem.

THEOREM 3. *Given a square ABCD, central with M, which lies in the xy plane of the trihedral. If the point A defines a congruence whose middle envelope is the center of S, so also does C, the opposite vertex, while the opposite vertices B and D define normal congruences; and the four points which bisect the sides of the square define four congruences with a common middle envelope.*

4. *C and $\bar{C}$ Each Normal.* Let C and $\bar{C}$ be normal congruences. Then*

$$a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad \bar{a} = e^{-\lambda}\frac{\partial \bar{P}}{\partial u}, \quad b = e^{-\lambda}\frac{\partial P}{\partial v}, \quad \bar{b} = e^{-\lambda}\frac{\partial \bar{P}}{\partial v}. \tag{4}$$

By (1) and (4), a necessary and sufficient condition that the congruences C and $\bar{C}$ have a common middle envelope is that

$$\begin{aligned} a - \bar{a} &= e^{-\lambda}\left(\frac{\partial P}{\partial u}-\frac{\partial \bar{P}}{\partial u}\right) = e^{-\lambda}\frac{\partial R}{\partial v}, \\ b - \bar{b} &= e^{-\lambda}\left(\frac{\partial P}{\partial v}-\frac{\partial \bar{P}}{\partial v}\right) = -e^{-\lambda}\frac{\partial R}{\partial u}. \end{aligned} \tag{5}$$

From (5), we have, from $\partial^2 R/\partial u\partial v=\partial^2 R/\partial v\partial u$, which is the condition of integrability,

$$\frac{\partial^2}{\partial u^2}(P-\bar{P})+\frac{\partial^2}{\partial v^2}(P-\bar{P}) = 0.$$

We have therefore the following theorem.

THEOREM 4. *A necessary and sufficient condition that the normal congruences (4) have a common middle envelope is that $(P-\bar{P})$ be a solution of Laplace's equation.*

WESLEYAN UNIVERSITY

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* Foster, loc. cit., p. 173.