ON A THEOREM OF PLESSNER*

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If $f(t)$ is a real-valued function of a real variable, periodic with period 2π and of bounded variation, then $f(t)$ is absolutely continuous provided $v(u)$, the total variation of $f(t+u)-f(t)$ on any interval of length 2π, tends to zero with u. This theorem, which is the converse of a well known theorem of Lebesgue, has been proved by Plessner, and by Wiener and Young.† Ursell‡ has given some interesting results concerning the total variation of $f(t+u)-f(t)$ for measurable functions $f(t)$, which when combined with the above theorem show (as he has pointed out) that that theorem holds for measurable functions. The papers referred to contain the essential ideas sufficient for a very short proof of the general theorem. In fact, with two additional lemmas which are given below, the proof as given by Plessner holds in the general case.

By an *admissible function* will be meant one which is finite-valued and periodic with period 2π.

THEOREM 1. *If $f(t)$ is admissible and $\lim_{u=0} v(u)=0$, then $v(u)$ is continuous.*

Let u_1, u_2 be any real numbers and $\delta_i=(t_{i-1}, t_i)$ be a partition of the interval $(-\pi, \pi)$. Then, if $t_i' = t_i+u_1$, the intervals $\delta_i' = (t_{i-1}', t_i')$ form a partition of $(-\pi+u_1, \pi+u_1)$ and§

$$\delta_i\{f(t+u_1+u_2)-f(t)\} = \delta_i\{f(t+u_1)-f(t)\} + \delta_i'\{f(t+u_2)-f(t)\},$$

and so $v(u_1+u_2)\leqq v(u_1)+v(u_2)$. This shows that v is finite everywhere; for if $v(u)<K$ on $(-a, a)$, then $v(u)<2K$ on $(-2a, 2a)$.

* Presented to the Society, April 20, 1935.

† Plessner, *Eine Kennzeichnung der totalstetigen Funktionen*, Journal für Mathematik, vol. 160 (1929), pp. 26–32. Wiener and Young, *The total variation of $g(x+h)-g(x)$*, Transactions of this Society, vol. 35 (1933), pp. 327–340.

‡ Ursell, *On the total variation of $f(t+\tau)-f(t)$*, Proceedings of the London Mathematical Society, (2), vol. 37 (1934), pp. 402–415.

§ If $\delta=(a, b)$, by $\delta f(t+u)$ is meant $f(b+u)-f(a+u)$.

From the above inequality it follows that $|v(u)-v(u_1)| \leqq v(u-u_1)+v(u_1-u)$, which shows that $v(u)$ is continuous.

LEMMA 1. *If the function $f(t)$ is admissible and measurable and $\lim_{u=0} v(u)=0$, then $f(t)$ is uniformly continuous.*

If not, there exists a positive number γ such that for every $\alpha>0$ there is an interval $\delta=(t, t')$ in $(-\pi, \pi)$ with $|t-t'|<\alpha$ and such that $|\delta f(t)|>\gamma$. There is an $\alpha_1>0$ such that $v(u)<\gamma/3$ if $|u| \leqq \alpha_1$. Let β be a positive number $<\alpha_1/2$ and $<\gamma/2$. Then by the theorem of Lusin $|f(t)-f(t')|<\beta$ for all t, t' in the interval $I=(-2\pi, 2\pi)$ except those in a set E_β with measure less than β, provided $|t-t'|$ is sufficiently small, say $<\alpha_2$. Fix an interval $\delta=(t, t')$ such that $|t-t'|<\alpha_2$ and $|\delta f(t)|>\gamma$. Then with this δ and any u such that $|u| \leqq \alpha_1$, we have

$$\gamma/3 > v(u) \geqq |\delta\{f(t+u)-f(t)\}| > \gamma - |\delta f(t+u)|.$$

If there exists a u with $|u| \leqq \alpha_1$ such that $t+u$, $t'+u$ are both in $I-E_\beta$, then the above inequality gives the contradiction $\gamma/3>\gamma-\beta>\gamma/2$. To see that such a u exists define a one-to-one correspondence between the intervals $\Delta=(t-\alpha_1, t+\alpha_1)$ and $\Delta'=(t'-\alpha_1, t'+\alpha_1)$ by the equations $x=t+u$, $x'=t'+u$. If E is a set in Δ, E' will denote the corresponding set in Δ'. Now if no such u exists $E_\beta \supset E_\beta\Delta+(\Delta-E_\beta\Delta)'$ and so

$$\begin{aligned}\alpha_1/2 > \beta > m(E_\beta) &\geqq m[E_\beta\Delta + (\Delta - E_\beta\Delta)'] \\ &\geqq [m(E_\beta\Delta) + m(\Delta - E_\beta\Delta)']/2 = \alpha_1,\end{aligned}$$

which again is a contradiction.

THEOREM 2. *If $f(t)$ is admissible and measurable and if also $\lim_{u=0} v(u)=0$, then $f(t)$ is absolutely continuous.*

The proof is entirely analogous to that of Plessner. By Lemma 1, $f(t)$ is summable and if $s_n'(t)$ is the first arithmetic mean of its Fourier series, then

$$s_n'(t) - f(t) = \int_{-\pi}^{+\pi} \{f(t+u) - f(t)\} F_n(u)du,$$

where $F_n(u)$ is Fejér's kernel; and thus the total variation

$$T[s_n'(t) - f(t)] \leqq \int_{-\pi}^{+\pi} v(u)F_n(u)du \to 0$$

by Fejér's theorem, since $v(u)$ is continuous and $v(0)=0$. For $\epsilon > 0$ there is an n_0 and a $\delta > 0$ such that

$$T[s_{n_0}'(t) - f(t)] < \epsilon/2, \quad \sum | \delta_i s_{n_0}'(t) | < \epsilon/2$$

for an arbitrary set $\{\delta_i\}$ of non-overlapping intervals with $\sum |\delta_i| < \delta$. Thus, for such a set of intervals,*

$$\sum | \delta_i f(t) | \leqq \sum | \delta_i \{f(t) - s_{n_0}'(t)\} | + \sum | \delta_i s_{n_0}'(t) | < \epsilon.$$

It might be pointed out that the theorem as stated is no less general than the corresponding theorem where $f(t)$ is finite and measurable on (a, b) and $v(u)$ is the total variation of $f(t+u) - f(t)$ on $(a, b-u)$.

The following example which is similar to one given by Ursell is interesting in connection with Theorem 2. Let $\{x_a\}$ be a Hamel† base with $x_1 = \pi$, so that every real number is expressible uniquely as a finite linear combination of elements of $\{x_a\}$ with rational coefficients. If $t = r_1\pi + \sum r_i x_i$ and $f(t) = r_1 - t/\pi$, then $f(t)$ is admissible and additive, that is, $f(t+u) = f(t) + f(u)$, and hence $v(u)$ is identically zero. Theorem 2 shows that $f(t)$ is not measurable and the example shows that the hypothesis of measurability is essential in Theorem 2 as well as in Lemma 1.

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* My attention is called, by Hans Lewy, to the fact that the use of Fourier series may be avoided by approximating to $f(t)$ by means of the absolutely continuous functions $(1/h)\int_0^h f(t+u)du$.

† Hamel, *Basis aller Zahlen und der unstetigen Lösungen der funktional Gleichungen* $F(x+y) = F(x) + F(y)$, Mathematische Annalen, vol. 60 (1905), p. 459.