NOTE ON THE EQUATION OF HEAT CONDUCTION*

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1. *Introduction.* In recent years, operational methods in mathematics have been a subject of much discussion. Heaviside, in his papers, by somewhat artificial methods, succeeded in solving a number of differential equations, especially those common to the electro-magnetic theory. The newer developments in operational calculus make no attempt to follow Heaviside's methods. More recent literature shows that the Fourier integral (Jeffreys,† Bush‡) plays an important role in these methods. In a thesis of Levinson,§ it was demonstrated that the Fourier transform could be employed to even better advantage than the Fourier integral.

In this note, it is proposed to quote the Fourier transform theorem of several variables, and apply a particular form of it to the solution of the equation for the flow of heat in three dimensions.

2. *Fourier Transform of Several Variables.* Here we shall consider a function of k real variables $F(x_1, \cdots, x_k)$ in a closed domain $a_\lambda \leqq x_\lambda \leqq b_\lambda$, $(\lambda = 1, \cdots, k)$, capable of taking on complex values. Consider an integral with infinite limits, such as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(x_1, \cdots, x_k) dx_1 \cdots dx_k.$$

Such an integral is said to be convergent if the limit of the integral

$$\lim_{A_\lambda \to \infty, B_\lambda \to \infty} \int_{-B_1}^{A_1} \cdots \int_{-B_k}^{A_k} F(x_1, \cdots, x_k) dx_1 \cdots dx_k, \qquad (\lambda = 1, \cdots, k),$$

* From a thesis presented for the degree Master of Science at the Massachusetts Institute of Technology, Oct. 30, 1934, under the title *Applications of the Fourier transform theorem.* Presented to the Society, December 28, 1934.

† H. Jeffreys, *Operational Methods in Mathematical Physics*, Cambridge Tract No. 23, 1931.

‡ V. Bush, *Operational Circuit Analysis*, 1929.

§ N. Levinson, *Applications of the Fourier integral*, master's thesis, May, 1934, Massachusetts Institute of Technology, unpublished as yet.

is determined. If, moreover,

$$\int_{-\infty}^{\infty} \left| F(x_1, \cdots, x_k) \right| dx_\lambda < \infty, \qquad (\lambda = 1, \cdots, k),$$

then $F(x_1, \cdots, x_k)$ is called absolutely integrable.

Let there be given a function $F(x_1, \cdots, x_k)$ with the following properties:

1. It has only a finite number of finite discontinuities.
2. It is piecewise continuous.
3. It is absolutely integrable and bounded.

Then under these hypotheses the Fourier integral of the function $F(x_1, \cdots, x_k)$ exists and is given by the following expression:

$$(2\pi)^k F(x_1, \cdots, x_k)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-i\sum_{\lambda=1}^{k} [u_\lambda (t_\lambda - x_\lambda)]} dt_1 du_1 \cdots dt_k du_k.$$

By rearranging this integral, the Fourier transform theorem is obtained. Denote the integral

$$\left(\frac{1}{2\pi}\right)^{k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(x_1, \cdots, x_k) e^{-i\sum_{\lambda=1}^{k} t_\lambda x_\lambda} dx_1 \cdots dx_k$$

by $g(t_1, \cdots, t_k)$. Then we have

$$(2\pi)^{k/2} F(x_1, \cdots, x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(t_1, \cdots, t_k) e^{i\sum_{\lambda=1}^{k} t_\lambda x_\lambda} dt_1 \cdots dt_k.$$

That is, $g(t_1, \cdots, t_k)$ is the transform of $F(x_1, \cdots, x_k)$ and, reciprocally, $F(x_1, \cdots, x_k)$ is the transform of $g(t_1, \cdots, t_k)$. This dual character of the Fourier transform theorem plays an important role in the solution of certain partial differential equations.

In the particular problem treated here, we make use of a special form of the transform theorem, commonly called the Mellin transform. To bring the Fourier transform into this form, consider first a function $F_1(x_1, \cdots, x_k)$ given by

$$F_1(x_1, \cdots, x_k) = e^{-\sum_{\lambda=1}^{k} c_\lambda x_\lambda} F(x_1, \cdots, x_k), \qquad (c_\lambda > 0).$$

Then the Fourier integral for $F_1(x_1, \cdots, x_k)$ becomes

$$\left(\frac{1}{2\pi}\right)^k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\sum_{\lambda=1}^k [u_\lambda(t_\lambda - x_\lambda) - ic_\lambda t_\lambda]} F_1(t_1, \cdots, t_k) dt_1 du_1 \cdots dt_k du_k.$$

Let $u_\lambda = v_\lambda + ic_\lambda$. Then $F_1(x_1, \cdots, x_k)$ is

$$\left(\frac{1}{2\pi}\right)^k \int_{-ic_1-\infty}^{-ic_1+\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\sum_{\lambda=1}^k [(v_\lambda + ic_\lambda) x_\lambda]} \psi dt_1 dv_1 \cdots dt_k dv_k,$$

where ψ is

$$F_1(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^k c_\lambda t_\lambda} e^{-i\sum_{\lambda=1}^k (v_\lambda + ic_\lambda) t_\lambda},$$

or the value of $F(x_1, \cdots, x_k)$ is

$$\left(\frac{1}{2\pi}\right)^k \int_{-ic_1-\infty}^{-ic_1+\infty} \int_{-\infty}^{\infty} \cdots$$

$$\int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-i\sum_{\lambda=1}^k v_\lambda(t_\lambda - x_\lambda)} dt_1 dv_1 \cdots dt_k dv_k.$$

Now put $p_\lambda = iv_\lambda$ and we obtain for $F(x_1, \cdots, x_k)$ the value

$$\left(\frac{1}{2\pi i}\right)^k \int_{-i\infty+c_1}^{i\infty+c_1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^k p_\lambda(t_\lambda - x_\lambda)} dt_1 dp_1 \cdots dt_k dp_k.$$

If this integral is split into two parts as the Fourier integral was, the Mellin transform is obtained. That is,

$$g(p_1, \cdots, p_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^k p_\lambda t_\lambda} dt_1 \cdots dt_k,$$

and

$$F(x_1, \cdots, x_k) = \left(\frac{1}{2\pi i}\right)^k \int_{-i\infty+c_1}^{i\infty+c_1}$$

$$\cdots \int_{-i\infty+c_k}^{i\infty+c_k} g(p_1, \cdots, p_k) e^{\sum_{\lambda=1}^k p_\lambda x_\lambda} dp_1 \cdots dp_k.$$

In physical problems we require oftentimes no knowledge of the past state of the system. That is, we are only interested in everything which takes place when t (time) $\geqq 0$. It is for this reason that it is of convenience to arrange coordinates such that

$F(x_1, \cdots, x_k)=0$ for all $x_\lambda < 0$. Then $g(p_1, \cdots, p_k)$ becomes under this restriction

$$\int_0^\infty \cdots \int_0^\infty F(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^{k} p_\lambda t_\lambda} dt_1 \cdots dt_k,$$

since the interval $(-\infty, 0)$ does not contribute anything to this transform.

There is just one more theorem which is of interest. Given $f_1(x_1, \cdots, x_k)$ and $f_2(x_1, \cdots, x_k)$ and their respective transforms $g_1(p_1, \cdots, p_k)$ and $g_2(p_1, \cdots, p_k)$, what is the transform of $g_1(p_1, \cdots, p_k) g_2(p_1, \cdots, p_k)$? This is answered by the faltung theorem. That is, if the function $f_1(x_1, \cdots, x_k)$ is the transform of $g_1(p_1, \cdots, p_k)$ and the function $f_2(x_1, \cdots, x_k)$ is the transform of $g_2(p_1, \cdots, p_k)$, then the transform of the function $g_1(p_1, \cdots, p_k)\, g_2(p_1, \cdots, p_k)$ is given by

$$f(x_1, \cdots, x_k) = \int_0^{x_1} \cdots \int_0^{x_k} f_1(y_1, \cdots, y_k) \cdot f_2(x_1 - y_1, \cdots, x_k - y_k) dy_1 \cdots dy_k.$$

We must remark that this theorem holds under the restriction that $f_2(x_1 - y_1, \cdots, x_k - y_k) = 0$ for $y_\lambda > x_\lambda$ and $f_1(y_1, \cdots, y_k) = 0$ for $y_\lambda < 0$.*

3. *Heat Flow in an Infinite Medium.* We propose to solve here the following linear parabolic differential equation

$$K\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = \frac{\partial V}{\partial t}. \tag{1}$$

Multiply equation (1) by $e^{-p_1 x - p_2 y - p_3 z}$ and integrate from 0 to ∞ over x, y, and z. Then we find by the transform theorem

$$\frac{\partial g}{\partial t} = K(p_1^2 + p_2^2 + p_3^2) g. \tag{2}$$

Thus we see that p_1, p_2, p_3 act as partial derivatives in the sense that p_1 acting on g is the transform of $\partial V / \partial x$. Similarly p_1^2

* See S. Bochner, *Vorlesungen über Fouriersche Integrale*, 1932, for a complete discussion of the Fourier transform.

acting on g is the transform of $\partial^2 V/\partial x^2$. Solving equation (2) for g we get

$$g = Ae^{kt(p_1^2+p_2^2+p_3^2)}, \tag{3}$$

where A is a function of p_1, p_2, p_3 to be determined.

For initial conditions we put the original temperature distribution equal to $f(x, y, z)$ when $t=0$. Moreover, without any lack of generality, we can limit the flow to the positive octant so that $V=0$ for $x<0$, $y<0$, $z<0$. Let $H(p_1, p_2, p_3)$ be the transform of $f(x, y, z)$. Then $A(p_1, p_2, p_3)=H(p_1, p_2, p_3)$ and

$$V = \left(\frac{1}{2\pi i}\right) \int_{-i\infty+c_1}^{i\infty+c_1} \int_{-i\infty+c_2}^{i\infty+c_2} \int_{-i\infty+c_3}^{i\infty+c_3} He^{k(p_1^2+p_2^2+p_3^2)t} \cdot e^{p_1x+p_2y+p_3z}dp_1dp_2dp_3. \tag{4}$$

To evaluate this expression, consider the transform of

$$e^{kt(p_1^2+p_2^2+p_3^2)}.$$

That is, we must evaluate

$$\left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1^2+p_1x/(kt))}dp_1 \int_{-i\infty+c_2}^{i\infty+c_2} e^{kt(p_2^2+p_2y/(kt))}dp_2 \cdot \int_{-i\infty+c_3}^{i\infty+c_3} e^{kt(p_3^2+p_3z/(kt))}dp_3.$$

This integral may be considered as the product of three integrals, for example,

$$\left(\frac{1}{2\pi i}\right) \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1^2+p_1x/(kt))}dp_1. \tag{5}$$

This may be written as

$$\frac{e^{-x^2/(4kt)}}{2\pi i} \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1+x/(2kt))^2}dp_1. \tag{6}$$

If we put $iu_1=(kt)^{1/2}(p_1+x/(2kt))$, then equation (5) becomes

$$\frac{e^{-x^2/(4kt)}}{2\pi(kt)^{1/2}} \int_{-\infty+iB}^{\infty+iB} e^{-u_1^2}du_1,$$

where the B's include variables which do not depend on our integration variable. The integral in equation (6) is taken along a path parallel to the x axis. This contour may be deformed and the integral can be taken along the entire x axis. Then equation (6) goes over into

$$\frac{e^{-x^2/(4kt)}}{2\pi(kt)^{1/2}}\int_{-\infty}^{\infty} e^{-u_1^2}du_1.$$

This integral is the error integral and its value is $\pi^{1/2}$. Hence the product of three such integrals is

$$\frac{e^{-(x^2+y^2+z^2)/(4kt)}}{(4\pi kt)^{3/2}}.$$

Applying the faltung theorem to equation (4), we find

$$V(x, y, z, t) = \left(\frac{1}{4\pi kt}\right)^{3/2}\int_0^x\int_0^y\int_0^z f(x-x_1, y-y_1, z-z_1) \cdot e^{-(x_1^2+y_1^2+z_1^2)/(4kt)}dx_1dy_1dz_1.$$

This is the standard solution of this problem, which is usually obtained by separating variables.*

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* E. Goursat, *Cours d'Analyse*, vol. 3, p. 107, 1927.