ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

103. Dr. E. W. Titt: *$(n-1)$-dimensional characteristic strips and Cauchy's problem for partial differential equations.*

In the present paper the author obtains a Cauchy-Kowalewsky system of first-order partial differential equations equivalent to a second-order partial differential equation in n independent variables. In the case of an analytic equation with analytic data this system of first-order equations solves Cauchy's problem for the second-order equation without the usual transformation of coordinates. The key to the Cauchy-Kowalewsky system is the conception of an $(n-1)$-dimensional characteristic strip. Thus in the case of two independent variables these characteristic strips are one-dimensional. They are in fact the strips of second order found by Goursat, *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Second Ordre*, Hermann, 1896. The idea of an $(n-1)$-dimensional characteristic strip is common to both the first and the second order non-linear partial differential equation. In both cases these strips find application in the problem of determining manifolds for which Cauchy's problem is indeterminate. (Received January 26, 1935.)

104. Mr. Garrett Birkhoff: *Non-commutative integration.*

A constructive theory of integration for functions of a line to a (generally non-commutative) "complete normed vector ring" V is developed, entirely analogous to the author's theory of integration of functions of a space with a Borel ring of measurable sets to a Banach space. The integral is immensely more powerful than the usual product integrals, and specializes to the Denjoy integral (without use of derivatives) in the case V is the real number system. (Received January 19, 1935.)

105. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

This paper gives new necessary and sufficient conditions that all of the solutions of a system of linear differential equations with almost periodic coefficients should be almost periodic. It also gives new sufficient conditions that a particular solution should be almost periodic. (Received January 24, 1935.)

106. Mr. F. J. Murray (National Research Fellow): *On linear transformations in* $\mathfrak{L}_p$, $p>1$.

In this paper we study closed linear transformations (not necessarily limited) in $\mathfrak{L}_p$, by the device exploited by J. von Neumann, for $\mathfrak{L}_2$, of considering them as defined by a closed linear manifold in the product space $\mathfrak{L}_p \times \mathfrak{L}_p$, with certain obvious properties. The relationships between the operations of forming an adjoint, closure in a space with lower index, and intersection with a space of higher index, are obtained. It is possible to define a symmetric relationship like the one which holds for Hermitian operators in $\mathfrak{L}_2$. This is discussed in terms of the above results. Finally, we consider the closure in $\mathfrak{L}_p$, $1<p<2$, of transformations in $\mathfrak{L}_2$. General results are obtained in terms of the adjoint. We then consider projections and more specific results are obtained in terms of the closure of the ranges. Examples are given to illustrate how these results may be applied to the study of a particular transformation and to show that certain results are the best obtainable. (Received January 26, 1935.)

107. Dr. W. T. Martin (National Research Fellow): *Linear difference equations with arbitrary real spans.*

In the first part of this paper we consider difference equations of the form (1) $\sum_{s=0}^{m} c_s(z)\ F(z+\delta_s)=G(z)$, in which the δ_s are any positive numbers and the $c_s(z)$ are analytic functions, asymptotically constant in a given sector. The methods developed by Bochner (Mathematische Zeitschrift, vol. 33 (1931), pp. 426–450) in his treatment of equations of the form (1), where the functions are functions of a real variable x, are applicable to the case where the functions are analytic. We first study the equation (1) with constant coefficients and, using the results thus obtained, we next treat by successive approximations the equation (1) with asymptotically constant coefficients. In the second part of the paper we study systems of the form (2) $\sum_{k=1}^{p}\sum_{s=0}^{m_{jk}} c_s^{jk}(z)\ F_k(z+\delta_s^{jk})=G_j(z)$, $j=1, \cdots, p$. A symbolic method of the sort developed by Bochner (Mathematische Annalen, vol. 104 (1931), pp. 579–587) and Carmichael (Transactions of this Society, vol. 35, (1933), pp. 1–28) is used in the treatment of the system (2) with constant coefficients. Again, a method of successive approximations is used to treat the system (2) with asymptotically constant coefficients. (Received January 26, 1935.)

108. Professor Cornelius Lanczos: *A new approximation method in solving linear differential equations with non-oscillating coefficients.*

The method presented is based on the following principles. (1) The Hamiltonian canonical form of a linear differential equation in connection with the canonical transformations permits us to separate the periodic or exponential part of the solution, reducing the problem to the determination of a mere amplitude factor (see Annalen der Physik, vol. 20 (1934), pp. 635–688). (2) This amplitude factor is a slowly changing function which can be efficiently approximated by a small number of powers of x, either ascending or descending, according to the given range. (3) The powers of x in the real domain

$x=0$ to $x=1$ are highly non-orthogonal, that is, x^{n+1} can be practically replaced by a linear combination of the lower powers. The Taylor series and the ordinary semi-convergent power series expansions approximate efficiently only in the neighborhood of a point. The method here developed distributes the error evenly over a large range and allows us to approximate by a small number of powers much more efficiently, in accordance with the ordinary approximations by means of orthogonal function systems. The coefficients of the expansion, however, are here determined by simple recurrence formulas. (Received January 26, 1935.)

109. Dr. S. E. Warschawski: *On the angular derivatives of univalent functions.*

Let R be a simply connected region in the w-plane, $w=u+iv$, whose boundary contains $w=0$. For $|v|\leqq a$, $a>0$, let $u=f(v)$, with $f(0)=f'(0)=0$, represent an analytic arc, and $\delta(v)\geqq 0$ be a continuous function such that (i) $\delta''(v)$ is bounded, (ii) $\int_{-a} \delta(v)\, dv/v^{n+1}$, $n\geqq 1$, exists, and (iii) the boundary of R in a neighborhood of $w=0$ is between the curves $u=f(v)-\delta(v)$ and $u=f(v)+\delta(v)$, the latter being inside R. We prove that if $z=z(w)$ maps R conformally on $|z|<1$, $\lim_{w\to 0} z^{(\nu)}(w)=z^{(\nu)}(0)$ exists for $1\leqq\nu\leqq n$ in any angle (*) $|\arg w|\leqq\alpha<\pi/2$ within R, and also $\binom{*}{*}$ $z(w)=z'(0)\, w+(1/2)z''(0)\, w^2+\cdots+(1/n!)z^{(n)}(0)\, w^n+r_n(w)w^n$, where $\lim_{w\to 0} r_n(w)=0$ in (*). If (i) is replaced by the hypothesis that $\delta(v)$ increases monotonically with $|v|$, the expansion $\binom{*}{*}$ holds in a wider region than (*). For $n=1$ the latter result extends a theorem of C. Carathéodory and its generalizations by C. Visser, J. G. van der Corput, and the author. (Received January 26, 1935.)

110. Mr. D. L. Netzorg: *On positive Cotes' numbers.*

For given $x_1, \cdots, x_n$, $(x_1<\cdots<x_n)$, there is a mechanical quadrature formula $\int_{-1}^{1} f(x)\, dx=c_1f(x_1)+\cdots+c_nf(x_n)$, valid for all polynomials of degree less than n. We assume that the c's (which are the Cotes' numbers corresponding to the x's) are all positive. This entails that one, at least, of the x's lie between any two successive roots of $Q(x)=P_m(x)+aP_{m-1}(x)$, where $2m\leqq n+1$, $P_m(x)$ is the mth Legendre polynomial, and a is real. Furthermore x_1 is less than q_1, the smallest root of $P_{m-1}(x)$, while x_n is greater than its largest root. As to the Cotes' numbers themselves we find that if $r_i<x_j<r_{i+1}<r_{i+2}$, then $1+r_i<H_1+\cdots+H_i<c_1+\cdots+c_j<H_1+\cdots+H_{i+1}<1+r_{i+2}$, where the r's are roots of $Q(x)$, and the H's are the Cotes' numbers corresponding to the r's. Furthermore, if $x_j<q_1$, then $c_1+\cdots+c_j<G_1$, where the G's are Cotes' numbers corresponding to the roots of $P_{m-1}(x)$. From these facts it is easy to prove a result due to Fejér, obtained by him in a different fashion (Mathematische Zeitschrift, vol. 37 (1933), pp. 287–309). For any positive weight function similar results hold. (Received January 28, 1935.)

111. Dr. A. C. Offord: *On Fourier transforms*, III.

This paper is a continuation of an earlier investigation (London Mathematical Society Proceedings, (2), vol. 38 (1934), pp. 197–216). A condition for a function $f(x)$ to have a Fourier transform $F(x)$ in the class $L^p(1\leqq p\leqq\infty)$ is given and an expression for $f(x)$ in terms of $F(x)$ is found. Conversely, a condition is obtained for a function $F(x)$ of $L^p(p>2)$ to be the Fourier transform in L^p of some function $f(x)$. (Received January 30, 1935.)

112. Professor Otto Szász: *Converse theorems of summability for Dirichlet series.*

This paper contains simplifications and generalizations of some "Tauberian" theorems for Dirichlet series. Let $F(t)=\sum_1^\infty c_\nu e^{-\lambda_\nu t}$ converge for $t>0$ and let the limit $\lim_{t\to 0}F(t)=s$ exist. We derive various sufficient conditions for the convergence (to the value s) of the series $\sum_1^\infty c_\nu$. These conditions are more general than those known before. An essential feature of our conditions is that the customary hypothesis $\lambda_{\nu+1}/\lambda_\nu\to 1$ is entirely eliminated. (Received February 19, 1935.)

113. Professor C. R. Adams and Dr. Hans Lewy: *On convergence in length.*

Let $f_n(x)$ $(n=1, 2, 3, \cdots)$ be a sequence of functions of bounded variation converging on $a\leqq x\leqq b$ to a limit function $f_0(x)$ of bounded variation; let the length in the sense of Peano [total variation] of f_n on $a\leqq x\leqq b$ be denoted by $L(f_n)[T(f_n)]$ $(n=0, 1, 2, \cdots)$; then f_n is said to converge in length [variation] to f_0 on (a, b) when and only when we have $L(f_n)\to L(f_0)$ $[T(f_n)\to T(f_0)]$. Concerning convergence in variation see Adams and Clarkson, this Bulletin, vol. 40 (1934), pp. 413–417. The present paper (i) shows that convergence in length implies convergence in variation (the converse is not true even when f_n, $(n=0, 1, 2, \cdots)$, is assumed absolutely continuous); (ii) examines the relationship between convergence in length and uniform convergence; (iii) shows that convergence in length, when f_0 is absolutely continuous, implies $T(f_n-f_0)\to 0$; (iv) proves that convergence in length is invariant under addition and multiplication of sequences when the limit functions are absolutely continuous; and (v) shows that convergence in length need not be invariant under addition of two sequences when both limit functions are merely continuous even though all approximating functions are absolutely continuous. (Received February 14, 1935.)

114. Professor J. L. Walsh: *Orthogonal polynomials and functions of the second kind.*

Let $n(z)$ be positive and continuous on the rectifiable Jordan curve C, let $\{p_k(z)\}$ be the polynomials normal and orthogonal on C with respect to $n(z)$, and let $\{q_k(t)\}$ denote the corresponding functions of the second kind: $q_k(t)\equiv\int_C[n(z)p_k(z)|dz|]/(t-z)$, t exterior to C. Let $f(z)$ be analytic in the closed annulus A bounded by C and C_R (Kreisbild for the mapping of the exterior of C). We have, interior to A, uniformly on any closed set interior to A, $f(z)=(1/(2\pi i))\sum_{k=0}^\infty p_k(z)\int_\Gamma f(t)\,q_k(t)\,dt-(1/(2\pi i))\sum_{k=0}^\infty q_k(z)\int_\Gamma f(t)\,p_k(t)\,dt$, where Γ is an arbitrary contour in A containing C in its interior. (Received February 11, 1935.)

115. Professor M. Fekete: *Some generalizations of Paley's theorems on Fourier series with positive coefficients.*

This paper contains various generalizations of recent theorems of Paley, where the conditions of positiveness of the coefficients is replaced by conditions of the type $a_n+\alpha_n\geqq 0$, $b_n+\beta_n\geqq 0$ where α_n, β_n are terms of series which are "slowly oscillating" in a certain sense. Necessary and sufficient conditions are obtained for uniform convergence of the Fourier series of a continuous function, or for uniform boundedness of partial sums of the Fourier series of a bounded function. (Received February 23, 1935.)

116. Professor Malcolm Foster: *Congruences with a common middle envelope.*

Relative to the moving trihedral on the unit sphere S, let (a, b) and $(\bar{a}, \bar{b})$ be two points in the tangent plane through which two lines l and $\bar{l}$ are drawn parallel to the normal to S. The necessary and sufficient condition that the congruences formed by the lines l and $\bar{l}$ have a common middle envelope is that the congruence $(a-\bar{a}, b-\bar{b})$ has the center of S for its middle envelope. The paper considers the case of a common middle envelope for two normal congruences, and for the congruences defined by the points (a, b) and $(-b, a)$. (Received January 25, 1935.)

117. Miss Jeanette Fox: *Finiteness of the number of quadratic fields with even discriminant and Euclid algorithm.* Preliminary report.

The only quadratic fields $R(m^{1/2})$, m even, which have a Euclid algorithm are $R(2^{1/2})$ and $R(6^{1/2})$. For quadratic fields $R(m^{1/2})$, $m\equiv 3 \pmod 4$, there are at most four fields with Euclid algorithm. Additional results have been obtained for fields with odd discriminant. (Received January 26, 1935.)

118. Dr. D. H. Lehmer: *On Lucas's test for the primality of Mersenne's numbers.*

In 1930 the writer proved as a by-product of a long discussion that the sufficient condition of Lucas for the primality of $2^{4k+1}-1$ is also necessary and equally applicable to $2^{4k-1}-1$. In fact, we may state that a necessary and sufficient condition for the primality of $N=2^n-1$ (n is odd) is that N divide the $(n-1)$st term of the sequence 4, 14, 194, $\cdots$ in which each term is the square of its predecessor minus 2. In 1932 A. E. Western proved this theorem by means of the theory of algebraic numbers. In this paper the writer presents a very elementary yet self-contained proof of this test for primality. (Received January 25, 1935.)

119. Professor I. J. Schoenberg: *Note on metric geometry.*

The problem of constructing simplexes with assigned lengths for the edges in (definite or indefinite) euclidean spaces and in spherical spaces, studied recently by Menger, Blumenthal, Wald, Coxeter, Todd, and Klanfer, is very simply solved by means of quadratic forms. Thus for the sphere $S_r^\rho: x_1^2+x_2^2+\cdots+x_{r+1}^2=\rho^2$ we have the following theorem. Let α_{ik}, $(i\neq k; i, k=1, \cdots, n; \alpha_{ik}=\alpha_{ki})$, be $n(n-1)/2$ given positive quantities. A necessary and sufficient condition that there be, on some spherical space of radius ρ, n points A_1, $A_2, \cdots, A_n$ of mutual (shortest) spherical distances $A_iA_k=\alpha_{ik}$, $(i\neq k)$, is that the quadratic form $F=\sum_{i,k=1}^{n} \cos (\alpha_{ik}/\rho)\ x_ix_k$, $(\alpha_{ii}=0)$, be always $\geqq 0$. If r is the rank of F, then we can find such points in S_{r-1}^{ρ} but not in $S^{\rho}{}_{-2}$. (Received January 28, 1935.)

120. Professor Oscar Zariski: *On the Poincaré group of plane algebraic curves.* First communication.

The paper deals with the Poincaré group G of the residual space of the dual C of any plane algebraic curve Γ with ordinary double points and of any other curve of which C is a limiting case. Use is made of the following lemma: *the Poincaré group of an algebraic hypersurface V in S_r with respect to its carrying S_r coincides with the Poincaré group of a generic plane section of V with respect to its carrying plane.* In the present communication the case where Γ is rational is considered. If Γ is of order n, C is the plane section of the discriminant hypersurface in an S_n. It is found that G possesses $n-1$ generators $g_1, g_2, \cdots, g_{n-1}$ (Hurwitz) and the generating relations are: (a) $g_ig_j=g_jg_i$, if $i-j\neq\pm 1$; (b) $g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1}$; (c) $\prod_{i=1}^{n-1} g_i\prod_{i=n-1}^{1} g_i=1$. It is then proved that any plane curve $\overline{C}$ which has C as a limiting case but has less or simpler singularities than C possesses a cyclic Poincaré group. In particular, *the Poincaré group of any rational curve of even order, other than C, is cyclic.* The case where C is a sextic is an exception. (Received February 23, 1935.)

121. Dr. C. C. Grove: *A note on the word mantissa.*

The statements in most dictionaries on this word led me to original sources and uses because there were some manifest misstatements. The word first appeared in mathematics in its true etymological sense in the Latin edition of the *Algebra* of Johannis Wallis, 1693, p. 41: Ejusque partes decimales abscissas, 0.12003416, *Appendicem* voco, sive *Mantissam fractionis primae*, seu *Mantissam primum.* This arose in writing out the successive convergents of the continued fraction for 2684769/837.6571 = 1/3.12003416+. Two chapters farther on he does not use the word in writing on logarithms. Leonardus Euler in his *Introductio in Analysin Infinitorum*, 1748, may conceivably have etymologised mantissa popularly as a Greek feminine of *μάντις* "prophet" in saying "Sola ergo Mantissa indicabit figuras numerum componentes." He thus introduced the use of the word in logarithms. Finally, C. F. Gauss in *Disquisitiones Arithmeticae*, 1801, goes back somewhat towards its original use, in Art. 312, p. 543: "Definitio. Si fractio communis in decimalem convertitus, seriem figurarum decimalium fractionis mantissam vocamus, ... Ita e.g. fractionis 1/8 mantissa est 125, mantissa fractionis 35/16, 1875, fractionis 2/37 mantissa 054054 $\cdots$ in inf." (Received February 14, 1935.)

122. Professor H. A. Simmons: *The classification and general solution of certain diophantine problems which involve special systems of equations of the second degree.*

The main result of the paper, expressed in Theorems 2 and 6, is essentially as follows: consider the mixed system of diophantine equations and inequalities T: $a_1(n)x_1 - a_j(n)x_j + b_j(n) = 0$, $(j = 2, \cdots, m)$, $x_1 \geqq x_2 \geqq \cdots \geqq x_m > 0$, $n > 0$, $a_1(n) \geqq a_2(n) \geqq \cdots \geqq a_m(n)$, where the $a_i(n)$, $(i = 1, \cdots, m)$, $b_j(n)$ are functions of n that take only positive integral values for positive integral values of n and the inequality sign holds between at least one pair of the a's (for every positive integer n). Suppose T is solvable when n is assigned a particular positive integral value, n_0, and let T_0 be the system gotten by setting $n = n_0$ in T. System T_0 has only a finite number of solutions. If also there is only a finite number of values of n for which T has a solution (as is the case in §3 of the paper), then T, considered as a system with n and the x's as unknowns, has only a finite number of solutions. Let T' be the system that is obtained from T by merely reversing the inequality signs between the a's. Then if n_0 is a value of n for which T' is solvable, T' has infinitely many solutions associated with n_0 (as is the case in certain examples of §4 of the paper). (Received February 6, 1935.)

123. Professor Raymond Garver: *On Bieberbach's theory of cubic constructions.*

In vol. 167 of the Journal für Mathematik, Bieberbach solves the problem of cubic constructions with the aid of a very simple use of a movable right angle. Constructions of this sort are, of course, known. The present paper studies Bieberbach's method of trisection, proves it equivalent to trisection with the aid of a limaçon, gives an alternate form not using a right angle but a different type of motion, and applies the method to certain particular constructions. (Received January 28, 1935.)

124. Professor C. N. Moore: *On convergence factors in multiple series summable by Nörlund means.*

The purpose of the present paper is to generalize to multiple series the theorem concerning convergence factors in simple series given in a paper presented at the annual meeting in Pittsburgh (see this Bulletin, abstract 41-1-56). The conditions obtained include the analogues of those obtained in the previous paper and certain supplementary conditions, such as usually arise in extending convergence factor theorems to multiple series. (Received January 26, 1935.)

125. Professor Edward Kasner: *A new inversive invariant.*

If we consider two curves touching each other (horn angle), a simple projective invariant is the ratio of the two curvatures γ_1 and γ_2 at the point of contact (Mehmke's theorem). The present author obtains a differential invariant for the inversion group, namely $I_3 = (d\gamma_1/ds_1 - d\gamma_2/ds_2)/(\gamma_1 - \gamma_2)^2$. This is a combination of the curvatures and the deviations (or rates of change of curvature with respect to arc length). This fundamental quantity I_3 is not only an *inversive* invariant, it is also a *conformal* invariant. (See Kasner, *Conformal geometry*, Cambridge International Mathematical Congress Proceedings, 1912, vol. 2, p. 84.) (Received January 15, 1935.)

126. Professor J. A. Shohat: *On mechanical quadratures.*

The author tries to approximate $F(h)=\int_a^{a+h} f(x)dx$ by an expression of the type $\Psi(h)=h\sum_{i=0}^{k=1} b_i f(a+n_i h)$, $0\leqq n_i\leqq 1$, where the constants b_i (independent of h) and n_i are so chosen that $F(h)-\Psi(h)=O(h^{2k+3})$. There is thus obtained a mechanical quadratures formula either of ordinary Gauss type or with one or both end abscissas fixed at the end points of the interval of integration. *The interior points turn out to coincide with the zeros of a certain class of Jacobi polynomials.* It is shown that the classical trapezoidal and Simpson formulas of mechanical quadratures are a special case, and thus their remainder is immediately derived. Some general properties of mechanical quadratures formulas with fixed abscissas are then discussed, and the asymptotic behavior of some of their coefficients for $n\rightarrow\infty$ is described. (Received January 17, 1935.)

127. Professor A. D. Campbell: *Pseudo-covariants in a Galois field and their algebraic and geometric applications.*

In this Bulletin for April, 1933, the author discussed briefly certain collections of terms from n-ics in m variables in a Galois field that have the peculiarity of being irreplaceable by any linear transformation if they are missing from the original n-ic. These sets of terms were called pseudo-covariants The present paper contains a more precise and thorough discussion of these pseudo-covariants as to their number for any n-ic, their exact definition, their order of disappearance. Also a study is made of the algebraic and geometrical meanings for the n-ic of the identical vanishing of these pseudo-covariants. (Received January 24, 1935.)

128. Mr. Max Astrachan: *Studies in the summability of Fourier series by Nörlund means.*

The author gives sets of sufficient conditions on the generating sequence $\{p_\nu\}$ in order that (I) the Fourier series and the conjugate Fourier series of $f(x)\subset L$ be summable by the Nörlund method (N, p_ν) to the correct value at all points where $\phi_1(t)=\int_0^t[f(x+s)+f(x-s)-2f(x)]\,ds=o(t)$, and $\int_0^t[f(x+s)-f(x-s)]\,ds=o(t)$, respectively; (II) the rth derived Fourier series be summable to the rth generalized derivative of $f(x)$ (in the sense of de la Vallée-Poussin) whenever this derivative exists; (III) the rth derived conjugate Fourier series of $f(x)$ be summable. The conditions, when (N, p_ν) is specialized to be (C, α), imply the restrictions that for (I) $\alpha>1$, (II) $\alpha>r$, (III) $\alpha>r+1$. If the method (N, p_ν) be superimposed on $(C, 1)$ the Fourier series is summable to $f(x)$ at all points where $\phi_1(t)=o(t)$ provided the generating sequence $\{p_\nu\}$ satisfies the Fourier-effective conditions of Hille-Tamarkin. (Received January 16, 1935.)

129. Mr. Garrett Birkhoff: *Abstract continuous groups.*

An axiomatic treatment of abstract continuous groups is given, based on properties of linear deformations of Banach space. The definition specializes to that of abstract Lie groups in case the infinitesimal generators have a finite basis, and to that of Banach spaces in case the groups are Abelian. The theory is then used as a tool in a general investigation into the structure of groups of deformations. (Received January 19, 1935.)

130. Dr. L. A. Dye: *Involutorial transformations associated with a rational ruled surface.* Part I.

In this part of the paper a (1, 1) correspondence is established between a pencil of quadric surfaces and the generators of a ruled surface R_{n+1} of order $n+1$ with an n-fold line l. A point P in space determines a quadric of the pencil and a generator r of R_{n+1}. The plane determined by P and r is tangent to R_{n+1} at a point Q. The line PQ meets the quadric in a residual point P' which is the image of P in an involutorial space transformation. Two cases are considered here: (1) the pencil of quadrics have a quartic space curve of the first species in common; (2) the pencil of quadrics have the line l and a space cubic as a basis curve. Some interesting contact conditions are involved in the homaloidal webs of surfaces arising from these transformations. (Received January 23, 1935.)

131. Professor Edward Kasner: *The dual of Lie's contact transformation of dynamics.*

The infinitesimal contact transformations of dynamics, first studied by Lie, have a characteristic function of the form $W=\Omega(x, y)\ (1+p^2)^{1/2}$. They may be characterized by the fact that the path of any element (x, y, p) is an orthogonal series of elements. A generalization introduced by the author arises by requiring the path series to be *isogonal* series. The ∞^2 path curves form a generalization of natural families. A dual type of infinitesimal contact transformation is obtained by demanding that the path series shall be *equitangential*. The characteristic function $W(u, v, w)$, where w is dv/du, is obtained in explicit form: $W=e^{-w/c}\ \Omega(u, v)$. The ∞^2 paths are studied; the ∞^2 envelope curves form a new type of family which may be regarded as the dual of the natural family. A further generalization is obtained by replacing the constant c by any function of the line coordinates (u, v). (Received January 15, 1935.)

132. Professor Edward Kasner: *Osculating turbines.*

The concept of geometric *turbine* was introduced by the author in a paper in the American Journal of Mathematics, 1911. A turbine consists of ∞^1 lineal elements whose points are on a circle and whose directions are equally inclined to the circle. In the present paper general *series* of ∞^1 elements are studied. At each element there is a definite osculating turbine. The set of ∞^1 osculating turbines defines a field, that is, a specific new type of differential equation of first order. The integral curves are termed the *related curves* of the given series. If a set of ∞^1 turbines is given, we obtain a criterion as to when it may be considered as an osculating set. The whole theory is a wide extension of the standard theory of osculating circles of curves. (Received January 16, 1935.)

133. Dr. Solomon Kullback: *A note on the analysis of variance.*

In a paper in this Bulletin, vol. 38 (1932), pp. 731–735, H. L. Rietz considers a set of independent items classified in some relevant manner into N sets of S items each. By making use of a dispersion theorem of Coolidge (this Bulletin, vol. 27 (1921), p. 439), Rietz arrives at estimates of variance used by

R. A. Fisher without making use of arguments involving the number of degrees of freedom of the items concerned. In this note we employ the same procedure for a set of independent items classified into N sets of $S_i (i=1, 2, \cdots, N)$ items each and arrive at estimates of variance used in discussions concerning the correlation ratio and multiple correlation coefficient. (Received January 19, 1935.)

134. Professor Marie Litzinger: *A basis for residual polynomials in n variables.*

Kempner established the existence of a basis for residual polynomials in one variable with respect to a composite modulus (Transactions of this Society, vol. 22 (1921), pp. 240–266), and considered the problem for n variables (Transactions of this Society, vol. 27 (1925), pp. 287–298). Dickson in his *Introduction to the Theory of Numbers*, pp. 21–27, developed a brief method of obtaining Kempner's results for polynomials in one variable. The purpose of this paper is the application of Dickson's method to polynomials in n variables. (Received January 22, 1935.)

135. Professor T. H. Rawles: *Operational methods in theoretical population problems.*

In this paper is discussed the state of a population which is subject to the following laws: (a) in each generation one member dies and another divides; (b) new members are of the same kind as the members whose division produced them. It is shown that the mean number of generations required for the entire population to reach a state where all members are the descendants of a single one of the original ones is $(N-1)^2$, where N is the number of members in the population. (Received January 19, 1935.)

136. Dr. W. T. Reid: *A certain three-dimensional continuum.*

In this note there is given an example of a bounded continuum in three-dimensional euclidean space such that there exists a point A of M and subcontinua G, K_1, and K_2 of $M-A$ satisfying the following conditions: (1) $M-K_i$, $(i=1, 2)$, is the sum of two mutually separated point sets each of which is connected, (2) each point of G is separated from A in M by either K_1 or K_2; however, (3) there does not exist a point set consisting of a finite number of connected sub-sets of M and separating A from G in M. (Received January 25, 1935.)

137. Dr. M. S. Robertson (National Research Fellow): *On the coefficients of a typically-real function.*

The class of functions $f(z)=\sum_1^\infty a_n z^n$ which are "typically-real" in the unit circle, that is, $f(z)$ is regular for $|z|<1$, a_n real, and $If(x+iy)>0$ when $y>0$ for $x^2+y^2<1$ (I denoting "the imaginary part of"), are considered. In particular, if $f(z)$ is regular, real on the real axis, and univalent for $|z|<1$, then $f(z)$ is also typically-real. It is shown that $E(r)\equiv\int_{-1}^{1} (f(rt)/t)dt$ is the greatest oscillation of the real part of the function $\int_0^z (f(z)/z)dz$ on $|z|=r$, and that $E=\lim_{r\to 1} E(r)$ exists, finite or infinite. If E is finite, then $|a_n|\leqq 2E/\pi$ for all n

and the equality sign is attained for n odd by the function $z/(1+z^2)$. (Received January 24, 1935.)

138. Professor Otto Szász: *Generalization of two theorems of Hardy and Littlewood on power series.*

Hardy and Littlewood have proved the following theorems: If $\lim_{x\to 1-0} \sum_0^\infty a_\nu x^\nu = s$ and (1) $na_n \geqq -k$, $k>0$, $n=1, 2, 3, \cdots$, then $\sum a_\nu$ converges to s; the same holds with the condition (2) $\sum \nu^\rho |a_\nu|^{\rho+1} < \infty$, $\rho>0$, instead of (1). The author replaces (1) and (2) respectively by $\sum_1^n \nu^p(|a_\nu| - a_\nu)^p = O(n)$ and $\sum \nu^\rho(|a_\nu| - a_\nu)^{\rho+1} < \infty$, giving simple proofs for these generalizations. (Received January 26, 1935.)

139. Mr. M. S. Webster: *The zeros of Jacobi polynomials.*

The main results of this paper may be divided into two parts. In the first part the author obtains, by a lengthy but elementary analysis, the number and location as far as the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ are concerned of the real zeros of the Jacobi polynomial $J_n(x; \alpha, \beta)$ where the parameters α, β are arbitrary real numbers (not necessarily positive). The second part, with α, β positive, contains upper and lower bounds for the said zeros which in this case are all inside the interval $(-1, 1)$. A geometric method employed by E. R. Neumann and C. Winston for Laguerre and Hermite polynomials is used. Applications are made to mechanical quadratures. (Received February 4, 1935.)

140. Professor C. G. Latimer: *On ideals in generalized quaternion algebras and Hermitian forms.*

The elements of a generalized quaternion algebra may be written in the form $x+yj$, where x, y are in a quadratic field F, $j^2=-\alpha$ a rational integer and $yj=jy'$, y' being the conjugate of y. Let G be the set of all such elements with x, y algebraic integers. It is shown that there is a one-to-one correspondence between certain classes of ideals in G and the classes of Hermitian forms over F of determinant $-\alpha$. Also there is proved a theorem on the existence of a g.c.d. and on factorization in G under the assumption that there is a single class of such forms. A number of results in the literature on special quaternion algebras may be deduced immediately from these results. There are also obtained for an infinitude of algebras the same theorems on the existence of a g.c.d. and on factorization in G as were obtained by Dickson for the Lipschitz integral quaternions (*Arithmetic of quaternions*, Proceedings of the London Mathematical Society, (2), vol. 20 (1922), pp. 225–232, Theorems 3, 8). (Received January 25, 1935.)

141. Dr. Solomon Kullback: *A note on the multiple correlation coefficient.*

In this note, there is derived the expected values of certain estimates of variance employed in R. A. Fisher's method of the analysis of variance of the multiple correlation coefficient. The results are established without making use of arguments involving the number of degrees of freedom of the items concerned. The method of procedure employs the theory of characteristic functions and makes use of an integral whose value has been established by S. S. Wilks. (Received February 5, 1935.)

142. Mr. H. M. MacNeille: *Extensions of partially ordered sets*. Preliminary report.

Elements $a, b, c, \cdots$, and a relation $\subset$ satisfying postulates (1) $a \subset b$ and $b \subset c$ imply $a \subset c$ and (2) $a \subset a$ form a partially ordered set. In terms of the given elements and relation, new relations, $\supset$ and $=$, are defined, and certain elements, not always present, are characterized as units, complements, sums, products, modular sums, modular products, distributive sums, and distributive products. Partially ordered sets are classified according to the characterized elements they contain. The classification includes lattices, complete lattices, modular lattices, distributive lattices, and Boolean algebras. The classifications, except those shown to be equivalent, are distinguished by examples. The equivalence of these definitions with the usual ones, for such classifications as Boolean algebras and the various types of lattices which are customarily defined in other ways, is established. The object of the extensions treated in this paper is to adjoin elements, as few as possible, to a given set, so that the resulting set belongs to a more specialized classification. The adjunction of units is immediate. Sums and products, of any number of elements, are adjoined by suitably defined cuts. Distributive sums or products are provided by adjoining ideals, and complements, by adjoining finite descending sequences. These extensions are established and applied. (Received February 2, 1935.)

143. Professor G. D. Birkhoff and Dr. M. R. Hestenes: *Oscillation, separation, and comparison theorems in the calculus of variations*.

We consider the accessory minimum problem described in abstract 41-3-173 and define the numbers m, r, s, as in that abstract. We suppose the strengthened condition of Legendre holds. The maximum number of admissible variations satisfying the end conditions and having $J_2(\eta)<0$ for every proper linear combination of these variations is the same for every such set and is equal to m. There are exactly $m-r$ conjugate points of x_1 between x_1 and x_2. For two different functionals J_2, J_2^* subject to different end conditions we always have $m-m^*=r-r^*+s-s^*$. Each conjugate family $\mathcal{J}$ belongs to some functional $J_2(\eta)$ and hence has an order of concavity r, namely that of $J_2(\eta)$. If the number of focal points of $\mathcal{J}$ between x_1 and x_2 is equal to m, then $m=r+s$. For two conjugate families $\mathcal{J}$, $\mathcal{J}^*$ we have $m-m^*=r-r^*$. The numbers r and r^* are easily computed and are each less than n, where $n+1$ is the dimensionality of our problem. From these results one can obtain as immediate corollaries previous oscillation, separation, and comparison theorems of this type. Applications have also been made to periodic extremals. We do not distinguish between the degenerate and non-degenerate cases. (Received February 18, 1935.)

144. Philip O. Bell: *Tetrahedra associated with canonical developments for the equation of a curved surface.*

A number of different canonical developments for the equation of a curved surface have been obtained by investigators in the field of projective differential geometry. These developments are simplest in form when the vertices of the associated tetrahedra are located on a certain quadric, known as the canonical quadric of Wilczynski. The geometrical location of this quadric was first accomplished by Wilczynski. Bompiani has offered a distinctly different definition, and more recently Stouffer has obtained a simple method of locating the quadric. The canonical quadric is actually useful in locating only one of the four vertices of the tetrahedron. It is the purpose of this paper to locate this fourth vertex for the whole series of expansions by rather elementary methods and without the introduction of Wilczynski's quadric. Incidentally, the quadric is located as soon as any one of these fourth vertices is determined. (Received March 2, 1935.)

145. Professor H. Bateman: *The problem of the shopper.*

Let $C(m, n)$ be the chance that at any shop a shopper spends n out of m cents in his possession. If he has p cents originally the chance $W(r, s)$ that he has r cents left after visiting s shops may be found by means of a recurrence relation. If, however, $G(r, t)$ is the generating function of the functions $W(r, s)$, a set of linear equations may be used for the determination of the functions $G(r, t)$. The recurrence relation may be solved also by means of a contour integral and in this paper a study is made of the functions which arise when special assumptions are made concerning the quantities $C(m, n)$. (Received March 5, 1935.)

146. Mr. D. L. Webb: *Generation of an n-valued logic by one binary operation.*

The author shows that any function of two elementary propositions in an n-valued logic can be generated by the iteration of one binary operation. This is accomplished by finding functions of two propositions in terms of this binary operation with propositions such that they can be combined by another binary operation, defined in terms of the first, so as to give any function of two elementary propositions defined by the truth table of an n-valued logic. (Received March 7, 1935.)

147. Mr. V. Elconin: *Primitive functions on a finite set.*

Let E be a set of n distinct elements, and let $p(u, v)$ be in E for any u, v in E. The "iterations" $f, g, h, \cdots$ of p are the functions of u or v or u, v with the property that any f is identically $p(g, h)$, or $p(g, b)$, or $p(a, h)$ or $p(a, b)$, for some g, h, a, b, where a, b are either u or v. Here $p(u, v)$ is "primitive" if and only if any function of u, v with values in E is an iteration of $p(u, v)$. It is shown in this note that the number of primitive functions is at least $n!$. From this it follows that the truth-functions of two variables in an n-valued logic can be generated from a single function, which is a generalization of Sheffer's well known result for $n=2$. (Received March 7, 1935.)

148. Professor A. D. Michal: *A reduced set of postulates for "Riemannian" differential geometry in abstract vector spaces.*

In two former papers (abstracts 39-11-332, 333) the author gave a set of postulates and developed a theory of parallel displacement and curvature for Riemannian differential geometry in Banach spaces. Since then several new theorems on differentials and total differential equations in abstract vector spaces have been proved. These theorems, among others, are used in the present paper to obtain a drastically reduced set of postulates. (Received March 7, 1935.)

149. Professor A. F. Moursund: *A note on Taylor's theorem.*

In this note, which is soon to appear in this Bulletin, we generalize the descriptive form of Taylor's theorem given by S. Pollard (Cambridge Philosophical Society Proceedings, Vol. 23 (1926–27), pp. 383–385). We consider only functions of one variable. (Received February 9, 1934.)

150. Professor Gordon Pall: *A new solution of the Gauss problem on $h(s^2d)/h(d)$.*

Let $h(d)$ denote the number of classes of primitive integral b.q.f.'s of discriminant $d\neq 0$. The formula $h(p^2d)/h(d)=\{p-(d|p)\}/\sigma$ is here proved by a new method, based on comparing the number of sets of representations of n and p^2n in the forms of discriminants d and p^2d. (Received February 11, 1935.)

151. Dr. J. M. Feld: *Configurations inscriptible in a plane cubic curve.*

This paper is concerned with the inscriptibility of certain configurations in real non-singular plane cubics, and with properties of these configurations. The configurations considered include the real Pappus configuration, a real configuration of twelve points and sixteen lines bearing the symbol 12_4, 16_3 and another comprising thirty-six points and eighty-four lines bearing the symbol 36_7, 84_3. (Received February 15, 1935.)

152. Professor E. T. Bell: *A detail in Kronecker's program.*

A misunderstanding regarding the freedom from circularity of Kronecker's method for eliminating all but the positive integers from analysis has recently been repeated. It is shown that the misunderstanding is removable, so far as negative numbers are concerned, by the correct form of the division transformation, not explicitly given by Kronecker, which is due to Wedderburn. Similar remarks apply to algebraic numbers. The paper will appear in the Journal of Philosophy of Science. (Received March 7, 1935.)

153. Mr. Edwin W. Paxson: *An abstract triplet metric space.* Preliminary report.

The usual concepts and postulates for vector spaces are taken with the replacement, however, of the set for the norm by a set of postulates for a function of three places. A trivial example is the space of ordered number-pairs in the plane, with the area of triangles as the function above, $N(x, y, z)$. With suitable definitions for N, the spaces of real continuous functions of real variables, and vector spaces, are other such systems. A triplet of elements is considered as the space unit of structure, and sequences of these are studied, as are functions on the triplet space to that space. (Received March 7, 1935.)

154. Miss Dorothy Manning: *On simple transitive permutation groups with transitive Abelian subgroups of the same degree.*

It has been shown by I. Schur that a simply transitive permutation group of degree n (n not a prime) in which there is a circular permutation of degree n is imprimitive and compound. It can also be proved that if a simply transitive permutation group G of degree p^{a+b} (p a prime) has a transitive Abelian subgroup of degree p^{a+b} and type (a, b), G is imprimitive and compound unless $a=b$. There exists a simply transitive primitive group of degree p^{2a} with a transitive Abelian subgroup of order p^{2a} and type (a, a) provided $p^{2a}>4$. (Received March 9, 1935.)

155. Professor H. F. Blichfeldt: *On geometry of numbers.*

The author has found that a slight modification of Winkowski's fundamental geometrical method gives surprising accuracy in certain applications. Special attention is given to the simultaneous approximation to two irrational numbers in rational fractions with common denominator, new lower limits being obtained. (Received March 9, 1935.)

156. Professor J. V. Uspensky: *A simple proof of Tschebysheff's inequalities.*

A method used in proving a certain fundamental lemma in the author's article, *On an expansion of the remainder in the Gaussian quadrature formula* (this Bulletin, vol. 40 (1934), p. 871) gives also a natural and simple proof of some important inequalities due to Tschebysheff. (Received March 9, 1935.)

157. Professor C. H. Forsyth: *A fallacy of the so-called osculatory interpolation formulas.*

Professor Forsyth shows that Newton's formula, which is usually the formula modified or used as a base to obtain the osculatory interpolation formula, is osculatory itself. (Received March 9, 1935.)

158. Mr. I. E. Highberg, Professor A. D. Michal, and Mr. A. E. Taylor: *The geometry of abstract euclidean spaces.*

An abstract euclidean space E (see A. D. Michal, abstract 40-11-383) is a normed linear space in which there is defined a Hermitian inner product (x, y), which is a continuous function on E^2 to C. This inner product is used to build up the *geometry* of E, (angle, distance, length of arc, etc.). The general theory of motions and rotations in E is considered. A rotation is shown to be additive, homogeneous of degree one for real multipliers, and, in case E is a complete space, continuous. The theory of Lebesgue integrals is discussed, together with its bearing on rectifiability of curves in E. For a Hilbert space or for an n-dimensional euclidean space, a necessary and sufficient condition that a curve be rectifiable is that its defining function be of bounded variation. (Received March 9, 1935.)

159. Mr. I. E. Highberg and Mr. A. E. Taylor: *Sets of independent postulates for linear spaces and vector spaces.* Preliminary report.

We have obtained a set of eleven postulates, defining a linear vector space closed under multiplication of reals. Equality is introduced on a postulational basis, and the reflexivity, symmetry, and transitivity properties follow as theorems. It is known that nine of these postulates are independent. Other systems of postulates are considered, dealing with linear spaces (non-metric) and with vector spaces closed under multiplication by complex numbers. In, these systems we have not stated the postulates for the multiplier domain (the system of real or complex numbers). In a later paper we intend to consider systems in which the multiplier domain is an abstract field, in which case the independence of all the postulates will be considered. (Received March 9, 1935.)

160. Mr. A. E. Taylor: *A set of independent postulates for a metric space.*

A metric space is herein defined by a set of six independent postulates. The equivalence relation in the space is a postulated relation, characterized by two of the six postulates. We are then able to show that it has the usual properties of equality. (Received March 9, 1935.)

161. Mr. Victor Elconin: *A theorem relating the differentials of Frêchet and Gateaux.*

Let $f(x)$ be in a Banach space for all x in some neighborhood of an element a in a Banach space E. It is well known that the Gateaux differential of $f(a)$ exists equal to the Fréchet differential of $f(a)$ if the latter exists. The theorem of this note states the converse of this proposition under the hypotheses that (i) some neighborhood of a is compact, and (ii) the Gateaux differential of $f(x)$ exists, continuous in x, linear in the increment of x, for all x in some neighborhood of a. Known results for n-space, Hilbert space, and certain function-spaces are shown to be instances of the theorem. (Received March 9, 1935.)

162. Dr. R. D. James and Mr. H. S. Zuckerman: *New results for the number $g(n)$ in Waring's problem.* Preliminary report.

The number $G(n)$ in Waring's problem is defined as the least value of s such that all *sufficiently large* integers are sums of s integral nth powers $\geqq 0$. The number $g(n)$ is defined as the least value of s such that *all* positive integers are sums of s integral nth powers $\geqq 0$. In the Comptes Rendus, Academie des Sciences, URSS, May, 1934, Vinogradow states that he has proved that $G(n) < 3k^2 + 4n$, k an integer, $k = n \log 8n + \theta$, $0 < \theta < 1$, and gives an outline of his proof. In this paper we supply a complete proof. At the same time we determine the value of C such that all integers $\geqq C$ can be expressed as a sum of $G(n)$ integral nth powers $\geqq 0$. As an illustration we consider 15th powers and are able to prove that $g(15)=33203$. This is the first exact value of $g(n)$ to be found since it was shown that $g(2)=4$ and $g(3)=9$. We hope to be able to prove a general result for $g(n)$. (Received March 9, 1935.)

163. Professor B. A. Bernstein: *Postulates for Boolean algebra involving the operation of complete disjunction.*

The operation of complete disjunction 0, given by $a0b=ab'+a'b$, has important properties. The author gives a set of ten postulates for Boolean algebra in terms of 0, $\times$, $'$, and obtains from these postulates some of the properties of 0. The postulates, like Sheffer's postulates in terms of "rejection," contain no postulated special elements, and no existence proposition other than one stating that the number of elements is at least two. A slight change in the postulates transforms them into a set of postulates for the logic of propositions. (Received March 9, 1935.)

164. Professor A. R. Williams: *Correspondences connected with a pencil of n-ics.*

When $n(n+3)/2-2$ base points of a pencil of n-ics are fixed in general position, and another describes a curve of order m, the locus of the remaining $(n-1)(n-2)/2$ is a curve of order $m(n^2-1)$ which has an mn-fold point at each of the base points. If $nq-(q-1)(q-2)/2$ of the fixed points are on a curve of order q, the number of variable points is $(n-q)(n+q-3)/2$, and their locus is a curve of order $m(n^2-q^2-1)$ which has an $m(n-q)$-fold point at each of the fixed points on the q-ic. Other features of the correspondence are discussed. (Received March 11, 1935.)

165. Professors H. E. Bray and G. C. Evans: *A simplification in the theory of weak convergence.*

By means of the correspondence between additive functions of point sets and additive functions of boundaries, the fundamental theorem on the weak convergence of a subsequence of a sequence of additive functions of point sets, uniformly bounded in total variation, may be reduced to the Cantor diagonal process for functions of finite plurisegments. (Received March 11, 1935.)

166. Professor Tibor Radó: *The isoperimetric inequality on the unit sphere.*

Let C be a simple closed curve of length L, located on the unit sphere, and let A denote the area of any one of the two regions bounded by C on the unit sphere. Then $(2\pi-A)^2-L^2\geqq 4\pi^2$, and the sign of equality holds if and only if C is a circle. F. Bernstein (Mathematische Annalen, vol. 60 (1905), pp. 117–136) proved this theorem under the restriction that the curve C can be covered by a hemisphere. The purpose of the present paper is to establish the theorem in complete generality. The generalization depends upon the following lemma: if the simple closed curve C, located on the unit sphere, has a length less than 2π, then C can be covered by an open hemisphere. (Received February 16, 1935.)

167. Professor I. S. and Dr. E. S. Sokolnikoff: *A Dirichlet problem for an ellipsoid lying between two infinite planes.*

A solution of the problem of determining the electrical field of a sphere lying between two infinite conducting planes was published by C. M. Rigby in Proceedings of the London Mathematical Society, (2), vol. 33 (1932). The present paper treats the more general problem in which the sphere is replaced by an ellipsoid of revolution. It contains Rigby's solution as a special case, and also enables one to calculate the field due to a charged circular disk or a thin rod placed between two parallel infinite conducting planes. The calculation of the harmonic function is reduced to the solution of an infinite system of linear algebraic equations, which are treated by a method of successive approximations. (Received March 1, 1935.)

168. Professor W. E. Roth: *On the characteristic values of the matrix $f(A, B)$.*

Let $A=A_1+A_2+\cdots+A_r$ be a given matrix of order n, where $|A_i-\lambda I| =(a_i-\lambda)^{n_i}$, $(i=1, 2, \cdots, r)$, $a_i\neq a_j$, $i\neq j$; and let $B=(B_{ij})$, where B_{ij} are $n_i\times n_j$ matrices. In order that every polynomial $f(A, B)$ in A and B have all its characteristic values in the set $f(a_i, b_j)$, where b_j are the characteristic values of B, then necessarily $|B|=\Pi_{i=1}^{r}|B_{ii}|$ and the characteristic value of B associated with a_i are those and only those of B_{ii}. If A is given in the Jordan canonical form, the matrix B is then displayed so that the polynomials $f(A, B)$ have the property above. The procedure of this paper is independent of that folllowed by Bruton. (Thesis, 1932, University of Wisconsin. Abstract, this Bulletin, No. 38-9-196.) (Received February 13, 1935.)

169. Dr. Ralph Hull (National Research Fellow): *Maximal orders in rational cyclic division algebras of odd prime degree.*

Let A be a cyclic division algebra of odd prime degree n over the field R of all rational numbers. Then A has a cyclic generation $A=(\sigma, Z, S)$, where Z is a cyclic field of degree n over R with a prime conductor $p\equiv 1 \pmod{n}$, discriminant p^{n-1}, and $\sigma=q_1\cdots q_s$ $(s\geqq 2)$, where the q are distinct primes, each an n-ic non-residue modulo p, but such that σ is an n-ic residue modulo p. (See abstract 40-3-128, March, 1934.) We have $A=Z+uZ+\cdots+u^{n-1}Z$, where $zu=uz^S$ for every z in Z and $u^n=\sigma$. There exists an integral quantity α of Z such that $N_{Z|R}(\alpha)=p^{n-1}$. It is proved that there are exactly n distinct maximal orders (integral sets) in A which contain all the integers of Z and the quantity u. These correspond to the n solutions λ of $\lambda^n\equiv\sigma \pmod{p}$ and for a fixed λ the corresponding maximal order has a basis y^iz_k $(i=0, \cdots, n-1; j=1, \cdots, n)$, where the z_k form a normal basis of the integers of Z and $y=(\lambda-u)\alpha/p$. These results are similar to those obtained by Albert (this Bulletin, vol. 40 (1934), pp. 164–176) for rational generalized quaternion algebras, but the methods used are more general, involving Hasse's theory of the invariants of cyclic algebra (Transactions of this Society, vol. 34 (1932), pp. 171–214), the well known representation of A as an algebra or matrices with elements in Z, and a study of the π-components of certain orders in A for all rational primes π. (Received March 7, 1935.)

170. Dr. Solomon Kullback: *On the distribution problem of statistics.*

We consider the problem of finding the distribution function (or probability law) of the functions $\mu_r(x_1, x_2, \cdots, x_n)$ $(r=1, 2, \cdots, p)$ of the variables $x_1, x_2, \cdots, x_n$ whose distribution function is given by the monotone absolutely additive set function $\phi(E)$ (where $F(x_1, x_2, \cdots, x_n)$ is the corresponding point function) such that $0 \leqq \phi(E) \leqq 1$ and $\phi(S)=1$ where S is the entire $(x_1x_2 \cdots x_n)$-space. The functions μ_r are real, single-valued, and the set of values of μ_r may be finite, a denumerable infinity, or form a continuum. By means of E. K. Haviland's extension to more than one dimension of P. Lévy's inversion formula for Fourier-Stieltjes transforms we are enabled to express the characteristic function of the distribution function of $\mu_1, \mu_2, \cdots, \mu_p$ as an integral of known functions over the entire $(x_1x_2 \cdots x_n)$-space. The theorems of this paper are generalizations of results previously obtained by C. V. L. Charlier, T. Kameda, and the author. (Received February 15, 1935.)

171. Dr. R. E. Basye: *Concerning an intrinsic property of plane continua.*

Let H and K be two mutually exclusive closed subsets of a compact plane continuum M. If for each point A of H and each point B of K there exists a finite number of subcontinua of M whose sum separates A from B in M, then there exists a finite number of subcontinua of M whose sum separates H from K in M. An example of a continuum showing that this result does not hold in three dimensiôns was obtained some years ago by W. T. Reid and presented by him at the meeting of the Society on February 23, 1935. (Received February 16, 1935.)

172. Dr. W. R. Thompson: *On criteria for rejection of observations and the distribution of the ratio of deviation to sample standard deviation.*

If one draws from a normal population a random sample of N observations $\{x_i\}$ whose mean is $\bar{x}$ and $N\cdot s^2=\sum(x_i-x)^2$, then it is proved in this paper that the distribution of $\tau\equiv(x-\bar{x})/s$ in repeated samples of the same size is given by substitution of $(n)^{1/2}\cdot z=t=(n)^{1/2}\cdot\tau/(n+1-\tau^2)^{1/2}$ in the z or t distribution of "Student" and R. A. Fisher, where $n=N-2$, i.e., $\tau=(n+1)^{1/2}\cdot\sin\alpha$, for $(n)^{1/2}\cdot\tan\alpha=t$. The probability, P, that $|\tau|>\tau_0$ is the same as that $|t|>(n)^{1/2}\cdot\tau_0/(n+1-\tau_0^2)^{1/2}$. Rejection criteria are readily established on a variety of bases, e.g., $P=\phi/N$, where ϕ is constant, and convenient tables are formed readily from those of "Student." Accordingly, all observations deviating by more than $s\cdot\tau_0$ from the sample mean are rejected. If one generalizes $\tau\equiv(\bar{x}_k-\bar{x})/s$, where $\bar{x}_k$ is the mean of k arbitrary elements of the sample of N, then it is proved that the distribution of τ is given by that of t by substitution of $t=\tau[(n\cdot k)/(n+2-k-k\cdot\tau^2)]^{1/2}$; i.e., $(k)^{1/2}\cdot\tau=(n+2-k)^{1/2}\cdot\sin\alpha$. (Received March 4, 1935.)

173. Professor G. D. Birkhoff and Dr. M. R. Hestenes:

Natural isoperimetric conditions for variable end point problems in the calculus of variations.

Consider the accessory minimum problem associated with the second variation $J_2(\eta)$ in the variable end point case. The minimum number m of natural isoperimetric conditions needed to reduce the arc $(\eta)\equiv(0)$ to a minimizing arc will be called the type number m of J_2. Let h be the maximum number of linearly independent accessory extremals satisfying the end conditions and having $J_2<0$ for every proper linear combination of these extremals. Let k be the maximum number of linearly independent accessory extremals having $\eta_i(x_1)=\eta_i(x_2)=0$ and which are such that no proper linear combination of these extremals satisfies the accessory transversality conditions. The number $r=h+k$ will be called the order of concavity of J_2. Let s denote the sum of the orders of the conjugate points of x_1 between x_1 and x_2. If the strengthened condition of Legendre holds, then $m=r+s$, even in the degenerate case. In the case of one variable end point, the type number m of J_2 is equal to the sum of the orders of the focal points between x_1 and x_2. In the general variable end point cases consider an extremal E satisfying the end conditions and the transversality conditions. By the type number m' of E will be meant the minimum number of isoperimetric conditions needed to reduce E to a weak minimizing arc. If the strengthened condition of Legendre holds and E is not degenerate, then $m'=m$, where m is the type number of J_2 on E. (Received February 18, 1935.)

174. Dr. Borge Jessen and Dr. Aurel Wintner: *Distribution functions and the Riemann zeta function.*

The paper starts with a systematic study of distribution functions in k-dimensional space and in particular of their infinite convolutions representing, in the language of the calculus of probability, the distributions arising by addition of an infinite number of independent random variables. The results are applied to almost periodic funtions and in particular to the Riemann zeta function. (Received February 26, 1935.)

175. Dr. L. M. Blumenthal (National Research Fellow): *Concerning metric transforms of metric spaces.*

Let $\phi(x)$ be a real function defined for every value $x=pq$, where pq is the distance of two points p, q, of a metric space M. We say that $\phi(x)$ is *metrically defined* over the space M. A space M' is called the *metric transform* of M by $\phi(x)$ provided (1) $\phi(x)$ is metrically defined over M, (2) the points of M and M' are in one-to-one, reciprocal, correspondence, and (3) if p', q' are points of M' corresponding to p, q of M, respectively, then $p'q'=\phi(pq)$. We write $M'=\phi(M)$. Consider the class Σ of monotonic increasing *concave* functions $\sigma(x)$ (i.e., for $x_1<x<x_2$, the three-rowed determinant in which the ith row is $\sigma(\xi_i)$, ξ_i, 1 and ξ_1, ξ_2, ξ_3 are respectively x_1, x, and x_2, is greater than zero) such that $\sigma(0)=0$. The following theorems are proved: Theorem 1. *If M is a metric space and $\sigma(x)$ is metrically defined over M, then $M'=\sigma(M)$ is a metric space.* Theorem 2. *If M is a metric space and $\sigma(x)=x^k$, $0\leqq k\leqq 1/2$, then $M'=\sigma(M)$ has each quad-*

ruple of its points congruent with four points of the three-dimensional euclidean space R_3. It is shown that $k=1/2$ is the greatest value of k for which Theorem 2 is valid. It is observed that the transform M' of a metric space M by a function $\sigma(x)$ of the class Σ is never *convex*. (Received February 21, 1935.)

176. Dr. J. F. Randolph: *Some density properties of point sets.*

The study of density functions began soon after the initial work on measure for point sets. Lebesgue himself showed that at almost all points of a Lebesgue measurable set the density of the set exists and is unity. The situation for the density functions of plane point sets in relation to Carathéodory linear measure is much more complicated, and considerable work has been done along this line. Much of this work has been concerned with the behavior of the density functions of a set at points of the set itself. R. L. Jeffery (*Sets of k-extent in n-dimensional space*, Transactions of this Society, vol. 35 (1933), pp. 629–647) considered the behavior of the density functions at points of the complement of the set as well, and obtained some enlightening results. In this paper, by entirely different methods, these results of Jeffery's are obtained and extended. Special cases of these results reveal new properties of Lebesgue measure for sets on a straight line. (Received March 5, 1935.)

177. Dr. Rufus Oldenburger: *Characteristic numbers of trilinear forms.*

Characteristic numbers of a trilinear form are defined by means of three-way determinants, which can be expanded according to various signancies of the indices giving as many as $3n$ characteristic numbers for a trilinear form of order n. Relations between the ranks of a trilinear form and the existence of characteristic numbers as well as their multiplicity are obtained. It is further shown that characteristic numbers are invariant under the class of non-singular linear transformations which leave unchanged a form of the type $x_iy_iz_i$, $i=1, 2, \cdots, n$. (Received March 9, 1935.)

178. Professor R. L. Wilder: *On locally connected spaces.*

Let S be a locally compact metric space that is locally i-connected, for $0\leqq i\leqq k$ (where k is a non-negative integer), in terms of Vietoris cycles and chains mod $m\geqq 2$. If F is a self-compact subset of S, and U an open subset of S containing F, then at most a finite number of i-cycles of F are independent with respect to homologies in U. As a corollary of this result, if S is compact, the Betti numbers $p^i(S, m)$ are all finite; and if in addition "small" $(k+1)$-cycles bound in S, then $p^{k+1}(S, m)$ is finite. Consequently if M is a g. c. n-m. (Annals of Mathematics, vol. 35, p. 878), all its Betti numbers are finite. Supplementary results include (1) a lemma characterizing absence of local i-connectedness (for $i=0$ proved by Moore; see this Bulletin, vol. 29, p. 296), and (2) the following theorem: In $E_n(n\geqq 2)$, let M be a compact continuum which is locally i-connected for $0\leqq i\leqq n-2$. Then if D is a domain complementary to M, the boundary of D is a Jordan continuum (for $n=2$, proved by M. Torhorst, Mathematische Zeitschrift, vol. 9, pp. 44–65). (Received March 9, 1935.)

179. Mr. H. E. Vaughan, Jr.: *Local Betti numbers.*

To each point a of a compact metric space M at which $\dim_a M = n$ a set of integers $\beta^i(a, M)$, $0 \leqq i \leqq n-1$, is assigned as follows: $\beta^i(a, M)$ is the least integer m, if it exists, such that for every $\epsilon > 0$ there exists a neighborhood of a of diameter less than ϵ whose boundary is $(n-1)$-dimensional and has m as its ith Betti number. If no such integer exists we define $\beta^i(a, M) = \omega$ or κ_0 in an obvious manner. Using these numbers we define *edge* and *kernel* points of M. The following characterizations are obtained: Let M be a 2-dimensional, locally 1-connected, closed cantorian manifold such that, if a is any point of M, $\beta^1(a, M) \leqq 1$. Then M is a 2-manifold. Let K be a 2-dimensional locally 1-connected irreducible membrane with respect to a 1-cycle carried by a simple closed curve J such that $p^1(K) = p^2(K) = 0$ and, if a is any point of K, $\beta^1(a, K) \leqq 1$ while if a is, in particular, a point of J, $\beta^1(a, K) = 0$. Then K is a closed 2-cell. By replacing the condition $p^1(K) = 0$ in the preceding theorem by $p^1(K) = 1$, a characterization of the Möbius strip is obtained. Similar changes give characterizations of other simple surfaces. (Received March 9, 1935.)

180. Professor G. D. Birkhoff and Dr. M. R. Hestenes: *Boundary value problems and the calculus of variations.*

Let $J(\eta, \sigma)$ be a quadratic functional in η_i, η_i' continuous in σ and subject to end conditions $\Psi_\mu(\eta, \sigma) = 0$ linear in the end values η_{i1}, η_{i2} and continuous in σ. This functional determines a boundary value problem. Suppose the strengthened condition of Legendre holds for all σ. If σ is a characteristic root let $d(\sigma)$ be its order. Otherwise set $d(\sigma) = 0$. Let $m(\sigma)$ be the type number of $J(\eta, \sigma)$ (see abstract 41-3-173). For σ near σ_0 we have $m(\sigma) \geqq m(\sigma_0)$, and if $d(\sigma) = 0$, $(\sigma \neq \sigma_0)$, then $m(\sigma_0) \leqq m(\sigma) \leqq m(\sigma_0) + d(\sigma_0)$. There are at least $|m(\sigma_2) - m(\sigma_1)|$ characteristic roots on the closed interval $\sigma_1\sigma_2$. If for every integer k there exists a constant σ_k and k arcs η_{ia} satisfying $\Psi_\mu(\eta, \sigma_k) = 0$ and having $J(\eta_a b_a, \sigma_k) < 0$ for every set $(b) \neq (0)$, then there exist infinitely many characteristic roots. This criterion is necessary in case $m(\sigma)$ is such that $m(\sigma) + d(\sigma) \leqq m(\sigma')$, $(\sigma < \sigma')$. In this case there are exactly $m(\sigma_2) - m(\sigma_1)$ characteristic roots on $\sigma_1 \leqq \sigma < \sigma_2$. If the end conditions are independent of σ and $J(\eta, \sigma) > J(\eta, \sigma')$ for $\sigma < \sigma'$ and $(\eta) \neq (0)$, then $m(\sigma) + d(\sigma) \leqq m(\sigma')$, $(\sigma < \sigma')$; this case is the one treated by Morse in his Colloquium Lectures, Chapter IV. These and further results are immediate consequences of the theory of natural isoperimetric conditions. (Received March 6, 1935.)

181. Professor R. L. Wilder: *On free subsets of E_n.*

Earlier results for the plane and ordinary 3-space by the same author (Fundamenta Mathematicae, vol. 21, pp. 156–167), relating to a question raised by Borsuk, are extended to higher dimensions. In $E_n (n > 2)$, let C be a free, compact continuum which cuts E_n, is locally i-connected for $0 \leqq i \leqq n-3$, and for which $p^1(C)$ (or $p^{m-2}(C)$) is finite. Then C is a generalized closed $(n-1)$-manifold (Annals of Mathematics, vol. 35, p. 878). As a locally contractible continuum is locally i-connected, an obvious corollary to the foregoing is obtained. When chains mod $m \geqq 2$ are employed, the conclusion of the above theorem follows if C is only assumed locally i-connected for $0 \leqq i \leqq (n-1)/2$, with no restriction on the Betti numbers. If a free, compact continuum which cuts E_n is deformable without meeting itself, then without further assumptions it is a g. c. $(n-1)$-m. Other theorems regarding particular types of transformations are also obtained. (Received March 9, 1935.)

182. Professor Walther Mayer: *The differential geometry of the submanifolds of the R_n of constant curvature.*

The present work is a new representation of the theory of curvature of submanifolds of spaces of constant curvature developed by C. Burstin and the author. (Received March 9, 1935.)

183. Dr. E. F. Beckenbach: *On subharmonic functions.*

It is shown that an equivalent form of a theorem of Montel and Radó is: if $p(u, v)$ is a positive continuous function, then a necessary and sufficient condition that $\log p(u, v)$ be subharmonic is that for every analytic function $f(u+iv)$ the function $p(u, v)|f(u+iv)|$ be subharmonic. Other equivalent forms are obtained. Applications are made to surfaces of negative curvature and to minimal surfaces. (Received March 7, 1935.)