DEGREE OF APPROXIMATION BY POLYNOMIALS TO CONTINUOUS FUNCTIONS*

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1. *Introduction.*† This paper is primarily concerned with functions analytic in a given region and continuous in the corresponding closed region; the region is in all cases bounded by an analytic Jordan curve with no double points. The existence of polynomials which converge uniformly to the function in the closed region has been established by Walsh‡ for more general regions, but it is frequently convenient to have more precise results in the study of functions on the boundary of the region. The method used here is essentially a reduction of the problem to the expansion of a function in a Fourier series and consequently the function is assumed to satisfy a Lipschitz or Hölder condition. An extension of the classical theory of Fourier series, identification of this expansion with the Taylor expansion on the circumference of the unit circle, and a study of the degree of approximation of Faber's polynomials belonging to the region form the basis of this investigation.

2. *Relation between the Fourier and Taylor Expansions.* The following theorem can be proved easily by separating the function into its real and imaginary parts and examining the coefficients.

THEOREM 1. *Let* $F(x)$ *be analytic in* $|x| < 1$, *continuous in* $|x| \leqq 1$. *Then the Taylor development for* $F(x)$ *about* $x = 0$ *is precisely the Fourier development for* $F(x)$ *on the circumference,* $|x| = 1$.

If $F(x)$, besides being continuous in the closed circle, satisfies

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‡ J. L. Walsh, Mathematische Annalen, vol. 96 (1926), pp. 430–436 and pp. 437–450.

a Hölder condition* of order α, $(0<\alpha\leqq 1)$, then the results on degree of approximation of Fourier series may be applied, where $F(x)$ is considered as a complex function of the real variable θ. It is necessary in this connection to show that $|F(x_1)-F(x_2)| < M|x_1-x_2|^\alpha$, $x_k=e^{i\theta_k}$, implies

$$|u(\theta_1) - u(\theta_2)| < M_1|\theta_1 - \theta_2|^\alpha, \quad |v(\theta_1) - v(\theta_2)| < M_1|\theta_1 - \theta_2|^\alpha,$$

where $F(e^{i\theta})=u(\theta)+iv(\theta)$, and this can be done without difficulty. *Also if* $|F^{(p)}(x_1)-F^{(p)}(x_2)| < M|x_1-x_2|^\alpha$, *where* p *is a positive integer or zero and* $F^{(p)}(x)$ *is the pth derivative (the zero-th derivative is the function itself), then the pth derivatives of* $u(\theta)$ *and* $v(\theta)$ *satisfy the corresponding conditions in* θ.

3. *Degree of Convergence of Fourier Series.*† A classical result in Fourier series is the following:

THEOREM 2. *If* $|f(x_1)-f(x_2)|\leqq M|x_1-x_2|$ *for all values of* x_1 *and* x_2, M *being a constant, then* $|f(x)-S_n(x)|\leqq (AM\log n)/n$, *where* A *is an absolute constant and* $S_n(x)$ *is the sum of the first* n *terms of the Fourier series for* $f(x)$.

This theorem in *real variables* can be extended‡ to include Hölder conditions of order α, $(0<\alpha<1)$. Now by an application of the results of §2 we get the following theorem.

THEOREM 3. *If* $F(x)$ *is analytic in* $|x|<1$, *continuous in* $|x|\leqq 1$, *and on the circumference* $|F(x_1)-F(x_2)|<M|x_1-x_2|$, *then*

$$|F(x) - P_n(x)| < (M_1 \log n)/n, \qquad |x| \leqq 1,$$

where $P_n(x)$ *is the sum of the first* n *terms of the Taylor development of* $F(x)$ *about* $x=0$.

Finally, by further application of the results on Fourier series and the relation to the Taylor development, we get the following theorem.

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* That is, $|F(x_1)-F(x_2)|<M|x_1-x_2|^\alpha$; here the Hölder condition will be used to include the case where $\alpha=1$.

† For the results used here on real variable approximation see Dunham Jackson, *Theory of Approximation*, 1930.

‡ See Jackson, loc. cit., pp. 4 ff. The proof goes through with slight modification for this case.

THEOREM 4. *If $F(x)$ is analytic in $|x|<1$, continuous in $|x|\leqq 1$, and $|F^{(p)}(x_1)-F^{(p)}(x_2)|\leqq M|x_1-x_2|^\alpha$, $(0<\alpha\leqq 1)$, where p is a positive integer or zero, then*

$$|F(x) - P_n(x)| \leqq (M_1 \log n)/n^{p+\alpha}, \qquad |x| \leqq 1.$$

4. *Faber's Polynomials.* Let C be an analytic Jordan curve in the x plane and let

$$(1) \quad x = \psi(t) = 1/t + a_0 + a_1 t + a_2 t^2 + \cdots = 1/t + \mathfrak{P}(t)$$

map the exterior of C on the region $|t|<r$ of the complex t plane, carrying $x=\infty$ into $t=0$. Due to the analyticity of C, the right-hand side of (1) converges for $|t|\leqq r'$, $r'>r$, and yields a uniquely inversible map. Denote $|t|=\rho<r'$ by K_ρ and let its image in the x plane be C_ρ; then $C_r=C$. Faber's polynomials* for the given region are defined as follows: $P_n(x)$ is the polynomial in x of degree n such that the coefficient of x^n is unity, and, as a function of t through (1), it has zero coefficients for the terms in $t^{-n+1}, t^{-n+2}, \cdots, t^{-1}, t^0$; hence $P_n(x)=1/t^n+t\mathfrak{P}_n(t)$, where $\mathfrak{P}_n(t)$ converges for $|t|\leqq r'$. Faber has proved that *any function analytic interior to C can be developed in a series, $\sum_{\nu=0}^\infty a_\nu P_\nu(x)$, which converges to the function at every interior point, and, conversely, every series which converges in the interior of C represents an analytic function.*

5. *Convergence to $F(x)$ in the Closed Region.* Now let $F(x)$ be an arbitrary function analytic in the region bounded by C_ρ, an arbitrary analytic Jordan curve, let the function be continuous in the corresponding closed region, and let $F(x)$ satisfy a Lipschitz condition (Hölder condition of order one) on C_ρ. Under these conditions the convergence of Faber's polynomials to $F(x)$ in the closed region will be established.

By Faber's results $F(x)=\sum_0^\infty a_\nu P_\nu(x)$, x within C_ρ. Since the curves are analytic, $X=\psi(\tau)$, where X is on C_ρ and τ is on K_ρ, is analytic and, consequently, $|F(X_1)-F(X_2)|\leqq A|X_1-X_2|$ implies

$$|F(\psi(\tau_1)) - F(\psi(\tau_2))| \leqq A\,|\psi(\tau_1) - \psi(\tau_2)|,$$

where $X_1=\psi(\tau_1)$, $X_2=\psi(\tau_2)$. But since $\psi(t)$ is analytic on K_ρ, $|\psi(\tau_1)-\psi(\tau_2)|\leqq B|\tau_1-\tau_2|$, and hence

* G. Faber, Mathematische Annalen, vol. 57, pp. 389–408; Journal für Mathematik, 1920.

$$|F(\psi(\tau_1)) - F(\psi(\tau_2))| \leqq AB|\tau_1 - \tau_2| = C|\tau_1 - \tau_2|.$$

Thus $F(\psi(t))$ satisfies a Lipschitz condition on K_ρ. From above,

$$F(\psi(t)) = \sum_0^\infty a_\nu(1/t^\nu + t\mathfrak{P}_\nu(t)), \qquad (\rho < |t| < r'),$$

where $|t\mathfrak{P}_\nu(t)| < G/r'^\nu$, G fixed; $a_\nu = [1/(2\pi i)]\int_{K_\rho} F(\psi(\tau))\tau^{\nu-1}d\tau$. We may write $F(\psi(t))$ in the form

$$(2) \qquad F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu + \sum_1^\infty b_\nu t^\nu, \qquad (\rho < |t| < r').$$

Since the right-hand side is analytic in $\rho < |t| < r'$, $\sum_1^\infty b_\nu t^\nu$ is analytic in $|t| < r'$. Now define $\phi(t) = -\sum_1^\infty b_\nu t^\nu + F(\psi(t))$. This function is analytic in $\rho < |t| < r'$, $F(\psi(t))$ satisfies a Lipschitz condition on $|t| = \rho$ and $-\sum_1^\infty b_\nu t^\nu$ is analytic in $|t| < r'$, and hence $\phi(t)$ satisfies a Lipschitz condition on $|t| = \rho$. From (2), $\phi(t) = \sum_0^\infty a_\nu/t^\nu$, $(\rho < |t| < r')$, and under the transformation $t = \rho/y$, $\phi(\rho/y) = \Phi(y)$, $\Phi(y)$ is represented by its Taylor development* in the ring $\rho/r' < |y| < 1$. The function $\Phi(y)$ may be defined throughout the unit circle by this convergent power series, and since $\Phi(y)$ is analytic throughout the unit circle, and satisfies a Lipschitz condition on the circumference, by Theorem 3 the Taylor development converges to $\Phi(y)$ on $|y| = 1$. This means that

$$-\sum_1^\infty b_\nu t^\nu + F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu, \qquad (\rho \leqq |t| < r');$$

and hence

$$F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu + \sum_1^\infty b_\nu t^\nu, \qquad (\rho \leqq |t| < r'),$$

which means *convergence in the closed region.*

By Theorem 3 we have $|\Phi(y) - \sum_0^n a_\nu y^\nu/\rho^\nu| < (M \log n)/n$ on and interior to $|y| = 1$. But $y = \rho/t$ and $\Phi(y) = \phi(\rho/y) = \phi(t) = F(\psi(t)) - \sum_1^\infty b_\nu t^\nu$, whence

$$\left| F(\psi(t)) - \sum_1^\infty b_\nu t^\nu - \sum_0^n a_\nu/t^\nu \right| < (M \log n)/n, \qquad (|t| = \rho),$$

* The original expression for a_ν under the transformation above serves to show that a_ν/ρ^ν is the νth Taylor coefficient of $\Phi(y)$.

which means that

$$\left| \sum_{n+1}^{\infty} a_\nu / t^\nu \right| < (M \log n)/n, \qquad (|t| = \rho).$$

But

$$\sum_1^\infty b_\nu t^\nu = \sum_1^\infty a_\nu t \mathfrak{P}_\nu(t) = \sum_1^n a_\nu t \mathfrak{P}_\nu(t) + \sum_{n+1}^\infty a_\nu t \mathfrak{P}_\nu(t),$$

and hence

$$\begin{aligned} F(\psi(t)) - \sum_1^\infty b_\nu t^\nu - \sum_0^n a_\nu / t^\nu &= F(\psi(t)) - \sum_1^\infty a_\nu t \mathfrak{P}_\nu(t) - \sum_0^n a_\nu / t^\nu \\ &= F(\psi(t)) - \sum_1^n a_\nu t \mathfrak{P}_\nu(t) - \sum_0^n a_\nu / t^\nu - \sum_{n+1}^\infty a_\nu t \mathfrak{P}_\nu(t) \\ &= F(\psi(t)) - \sum_0^n a_\nu (1/t^\nu + t \mathfrak{P}_\nu(t)) - \sum_{n+1}^\infty a_\nu t \mathfrak{P}_\nu(t). \end{aligned}$$

Thus

$$\begin{aligned} \left| F(\psi(t)) - \sum_0^n a_\nu (1/t^\nu + t\mathfrak{P}_\nu(t)) \right| &\leqq \left| \sum_{n+1}^\infty a_\nu / t^\nu \right| + \left| \sum_{n+1}^\infty a_\nu t \mathfrak{P}_\nu(t) \right| \\ &\leqq (M \log n)/n + \sum_{n+1}^\infty |a_\nu| \, |t \mathfrak{P}_\nu(t)|. \end{aligned} \tag{3}$$

But $|t\mathfrak{P}_\nu(t)| < G/r'^\nu$, G fixed, $\rho < r'$, $|t| < r'$, and

$$|a_\nu| = \left| \frac{1}{2\pi i} \int_{K_\rho} F(\psi(\tau)) \tau^{\nu-1} d\tau \right| \leqq 2\pi\rho M \rho^{\nu-1}/(2\pi) = M\rho^\nu.$$

Since $F(x)$ satisfies a Lipschitz condition on C_ρ, it is bounded there and thus $F(\psi(t))$ is bounded on K_ρ. Hence

$$\begin{aligned} \sum_{n+1}^\infty |a_\nu| \, |t\mathfrak{P}_\nu(t)| &\leqq \sum_{n+1}^\infty M\rho^\nu G/r'^\nu = \sum_{n+1}^\infty MG(\rho/r')^\nu \\ &= MG \sum_{n+1}^\infty (\rho/r')^\nu. \end{aligned} \tag{4}$$

But the remainder of a geometric series with quotient less than unity is of order less than $(\log n)/n$.* In consideration of this fact, an application of (4) to (3) yields

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* This follows directly from the fact that nx^{n+1} for $|x| < 1$ is the nth term of a convergent series. We shall use the fact that this geometric remainder is less than $(\log n)/n^\alpha$, $(0 < \alpha < 1)$, in a later proof.

$$(5) \qquad \left| F(\psi(t)) - \sum_0^n a_\nu(1/t^\nu + t\mathfrak{P}_\nu(t)) \right| \leqq (A' \log n)/n, \quad (t \text{ on } K_\rho).$$

The constant A' can be adjusted so that this inequality holds for all n. Since $x=\psi(t)$ and $P_\nu(x)=1/t^\nu+t\mathfrak{P}_\nu(t)$, (4) becomes

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n.$$

The result can be stated as follows.

THEOREM 5. *If $F(x)$ is analytic interior to the analytic Jordan curve C_ρ, continuous in the closed region, and satisfies a Lipschitz condition on C_ρ, then*

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n,$$

x on or within C_ρ, where A' is a constant independent of x and n (A' depends on $F(x)$ and on C_ρ) and $P_\nu(x)$ is the Faber polynomial of degree ν belonging to the region.

6. *Extension to Hölder Conditions and Derivatives.* Suppose $F(x)$ satisfies on C_ρ a Hölder condition of order α, $(0<\alpha\leqq 1)$, with the remaining hypotheses unchanged. By examination of the above it is seen that the first modification will come in (2) and by Theorem 4 the inequality will be

$$\left| \Phi(y) - \sum_0^n a_\nu y^\nu/\rho^\nu \right| < (M \log n)/n^\alpha.$$

The geometric remainder can be handled as before,* and we get the following theorem.

THEOREM 6. *If $F(x)$ is analytic interior to C_ρ, continuous in the corresponding closed region, and if*

$$|F(X_1) - F(X_2)| \leqq M |X_1 - X_2|^\alpha, \qquad (0 < \alpha \leqq 1),$$

X_1, X_2 on C_ρ, then

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n^\alpha, \ x \text{ on or within } C_\rho.$$

* See preceding footnote.

Now suppose, we have, instead of a Lipschitz condition on the function, $|F^{(p)}(X_1)-F^{(p)}(X_2)|\leqq M|X_1-X_2|^\alpha$, $(0<\alpha\leqq 1)$, p a positive integer. Let $F(x)=F(\psi(t))=f(t)$ and $t=\phi(x)$. Then $dt=\phi'(x)dx$, $dt/dx=\phi'(x)$. Since $F(x)$ has a pth derivative on C_ρ, $f(t)$ has a pth derivative on K_ρ. We have

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\begin{aligned}
F'(x) &= \frac{dF}{dx} = \frac{df}{dt}\frac{dt}{dx} = f'(t)\phi'(t),\\
F''(x) &= \frac{d}{dx}\frac{dF}{dx} = \frac{d}{dx}\left[f'(t)\phi'(x)\right] = f'(t)\phi''(x) + \phi'(x)f''(t)\phi'(x)\\
&= f'(t)\phi''(x) + [\phi'(x)]^2 f''(t),\\
F'''(x) &= \phi'''(x)f'(t) + \phi''(x)\phi'(x)f''(t)\\
&\quad + 2[\phi'(x)][\phi''(x)]f''(t) + [\phi'(x)]^3 f'''(t), \text{ etc.}
\end{aligned}
$$

Now, since $\phi(x)$ is analytic, all its derivatives satisfy Lipschitz conditions on C_ρ and hence with respect to t on K_ρ, and since $f(t)$ has a pth derivative on K_ρ its derivatives of all orders less than p satisfy Lipschitz conditions there, and hence it is seen that $f^{(p)}(t)$ satisfies the corresponding Hölder condition on K_ρ. The geometric remainder can be treated just as above, and thus we have the following theorem.

THEOREM 7. *Let $F(x)$ be analytic in the region bounded by the analytic Jordan curve C_ρ, continuous in the corresponding closed region, and further let*

$$|F^{(p)}(X_1) - F^{(p)}(X_2)| \leqq M|X_1 - X_2|^\alpha, \qquad (0 < \alpha \leqq 1),$$

X_1, X_2 on C_ρ, p a positive integer or zero; then

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (M_1 \log n)/n^{p+\alpha}, \ x \textit{ on or within } C_\rho.$$

This theorem includes all the preceding results as special cases.

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