Hence by 9, $(pq')^* \dashv [(pq')^* Z^*]$ in T.
Hence by 20 and 19, $(pq')^* \dashv \{[p(pq)']^*[(pq)p']^*\}$ in T.
Hence by 9, $(p \dashv q) \dashv [(p \dashv pq)(pq \dashv p)]$ in T.

23.2. $[(p \dashv pq)(pq \dashv p)] \dashv (p \dashv q)$ in T.

Proof. By 9 and 6, $(pp')^*$ in T, whence by 16, Z^* in T.
Hence by 19, when we replace p by $(pq')^*$ and q by Z^*,

$$\{[(pq')^* Z^*](pq')^{*\prime}\}^* \text{ in } T.$$

Hence by 9, $[(pq')^* Z^*] \dashv (pq')^*$ in T.
Hence by 20 and 19, $\{[p(pq)']^*[(pq)p']^*\} \dashv (pq')^*$ in T.
Hence by 9, $[(p \dashv pq)(pq \dashv p)] \dashv (p \dashv q)$ in T.

It will be noted that these proofs could be written out without the use of the "star" notation, since p^* serves merely as an abbreviation for $p \dashv p'$, and $(pq')^*$ as an abbreviation for $p \dashv q$.

Harvard University

ON INTEGRAL INVARIANTS OF NON-HOLONOMIC DYNAMICAL SYSTEMS†

BY A. E. TAYLOR

1. *Introduction.* It is well known that there are certain integral invariants associated with holonomic dynamical systems. Cartan‡ demonstrated that it is possible to characterize a Hamiltonian system by means of the relative integral invariant

$$\int_C \sum_i p_i dq_i - H dt.$$

The purpose of this paper is to extend the theory to the case of non-holonomic systems.

We shall adopt the following conventions in notation. There are three ranges of indices, which we shall usually represent by

† Presented to the Society, June, 20, 1934. I wish to acknowledge my indebtedness to A. D. Michal for criticism and suggestions during the writing of this paper.

‡ E. Cartan, *Leçons sur les Invariants Intégraux*, 1922, p. 13. Also W. F. Osgood, this Bulletin, vol. 39 (1933), p. 882, Abstract No. 343.

s, i, ν, $(s=1, 2, \cdots, n+k; i=1, 2, \cdots, n; \nu=1, 2, \cdots, k)$. Hereafter we shall not state explicitly these ranges for the indices.

2. *The Equations of Motion.* Consider a non-holonomic dynamical system with n degrees of freedom and $n+k$ natural coordinates $q_1, \cdots, q_{n+k}$, subject to the non-integrable relations

$$\dot{q}_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\dot{q}_i + a_\nu, \tag{1}$$

where the a's are continuous functions of $q_1, \cdots, q_{n+k}, t$, possessing continuous first partial derivatives. For such a system, assumed conservative, the equations of motion are*

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial L}{\partial q_i} - \sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu, && (i = 1, \cdots, n), \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{n+\nu}} &= \frac{\partial L}{\partial q_{n+\nu}} + \lambda_\nu, && (\nu = 1, \cdots, k), \end{aligned} \tag{2}$$

where the λ's are Lagrangian multipliers, functions of the time. It is to be observed that in case the system is holonomic all the λ's are zero, and Lagrange's equations hold in the usual form.

If we introduce new coordinates (q_s, p_s, t), where $p_s = \partial L/\partial \dot{q}_s$, and set

$$H = \sum_{s=1}^{n+k} p_s\dot{q}_s - L, \tag{3}$$

then equations (2) become†

$$\begin{aligned} \frac{dq_s}{dt} &= \frac{\partial H}{\partial p_s}, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} - \sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu, \\ \frac{dp_{n+\nu}}{dt} &= -\frac{\partial H}{\partial q_{n+\nu}} + \lambda_\nu, \end{aligned} \tag{4}$$

* P. Woronetz, *Über die Bewegung eines starren Körpers*, Mathematische Annalen, (1911), p. 421; also Whittaker, *Analytical Dynamics*, 1927, p. 214.

† For the transformation process, see Appell, *Traité de Mécanique Rationnelle*, vol. 2, 1911, p. 403.

and we find, in addition, that

$$(5)\qquad \frac{dH}{dt} = \sum_{\nu=1}^{k} a_\nu \lambda_\nu - \frac{\partial L}{\partial t}, \qquad \frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}.$$

3. *Integral Invariants.** Consider a system of differential equations

$$(6)\qquad \frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, t),$$

where the X's are continuous, together with all of their partial derivatives of the first order, in the neighborhood of the point $(x_1^0, \cdots, x_n^0, 0)$, and where not all the X's vanish at this point. Then we can write the solution in the form

$$(7)\qquad x_i = f_i(t; x^0, \cdots, x_n^0).$$

Let the initial values be made to depend upon a parameter:

$$x_i^0 = x_i^0(\alpha), \qquad (\alpha_0 \leqq \alpha \leqq \alpha_1),$$

these functions being continuous, with continuous first derivatives, such that

$$\sum_{i=1}^{n} \left(\frac{dx_i^0}{d\alpha}\right)^2 > 0.$$

Putting these values in (7), we get equations of the form

$$x_i = F_i(t, \alpha) = f_i[t; x_1^0(\alpha), \cdots, x_n^0(\alpha)],$$

and these equations represent a *tube* of trajectories of the given system (6). If we envisage a regular curve C encircling this tube, in such manner that each trajectory cuts C once and only once, in a point for which $t>0$, then we may speak of C as a *simple, closed circuit* of the tube of trajectories.

Now let functions $A_i(x_1, \cdots, x_n, t)$ and $A(x_1, \cdots, x_n, t)$ be given, and consider

$$(8)\qquad \int_C \sum_{i=1}^{n} A_i dx_i + A dt \equiv \int_{\alpha_0}^{\alpha_1} \left[\sum_{i=1}^{n} A_i \frac{\partial x_i}{\partial \alpha} + A \frac{\partial t}{\partial \alpha}\right] d\alpha.$$

If this integral has the same value for all such simple closed

---

* For other forms of definition, see Appell, loc. cit., vol. 2, p. 464.

circuits C encircling an arbitrarily chosen *tube* of trajectories of (6), it is called a *relative* integral invariant of the system (6). If, in particular, the circuit corresponds to a fixed instant t, then the integral has the form

$$\int_C \sum_{i=1}^{n} A_i dx_i \equiv \int_{\alpha_0}^{\alpha_1} \sum_{i=1}^{n} A_i \frac{\partial x_i}{\partial \alpha} d\alpha.$$

This we shall call a Poincaré* integral invariant, while (8) will be referred to as a Cartan invariant.

4. *The Integral Invariant.* Let there be given a non-holonomic system, subject to the kinematical conditions (1), and having the equations of motion (2). These two sets of equations define the q's and λ's as functions of the time. There is a $2(n+k)$-parameter family of path curves of the system, represented by the functions

$$(9) \qquad q_s = q_s(t; \overset{0}{q_1}, \cdots, \overset{0}{q}_{n+k}; \overset{\cdot 0}{q_1}, \cdots, \overset{\cdot 0}{q}_{n+k}),$$

where the $2(n+k)$ constants $(q^0, \dot{q}^0)$ are given initial values.

Let these path curves be interpreted in the space of the variables $(q_1, \cdots, q_{n+k}, \dot{q}_1, \cdots, \dot{q}_{n+k}, t)$ and consider an arbitrary regular curve† C_0 in the hyperplane $t=0$:

$$C_0: \qquad q_s^0 = q_s^0(\alpha), \qquad \dot{q}_s^0 = \dot{q}_s^0(\alpha), \qquad (\alpha_0 \leqq \alpha \leqq \alpha_1).$$

The tube of path curves emanating from the curve C_0 is the locus defined by the equations

$$(10) \qquad q_s = q_s(t, \alpha), \qquad \dot{q}_s = \dot{q}_s(t, \alpha),$$

obtained from (9) by means of the equations of C_0.

Now consider the action integral in the form

$$(11) \qquad J = \int_0^{t_1} L dt,$$

where t_1 is a variable upper limit. When this integral is extended along a path curve on the tube, it becomes a function of α. Let the aggregate of values of t_1 correspond to a simple closed circuit C of the tube. We introduce a new variable u, so that if

* H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, 1899, vol. 3, p. 4.

† It is sufficient to require that these functions be continuous, admitting continuous first derivatives, not all of which vanish simultaneously.

(12a) $$t = t(u, \alpha),$$

then

(12b) $$t(u_0, \alpha) = 0, \qquad t(u_1, \alpha) = t_1, \qquad \frac{dt}{du} = \frac{\partial t(u, \alpha)}{\partial u} = \rho > 0,$$

for all values of α, where u_1 and u_0 are constants. This can be done in a variety of ways, so that ρ may be taken arbitrarily,* within certain limits.

The integral (11) now becomes

(13) $$J(\alpha) = \int_{u_0}^{u_1} F du,$$

where

$$F(q_1, \cdots, q_{n+k}; q_1', \cdots, q_{n+k}', t', t)$$
$$= L\left(q_1, \cdots, q_{n+k}; \frac{q_1'}{t'}, \cdots, \frac{q_{n+k}'}{t'}, t\right)t',$$
$$t' = \frac{dt}{du}, \qquad q_s' = \frac{\partial q_s}{\partial u} = \dot{q}_s t'.$$

We next compute† $J'(\alpha)$, using Leibniz's rule, which is permissible since we have all the desired continuity. Upon carrying out the differentiation, and integrating by parts, we find

$$J'(\alpha) = \int_{u_0}^{u_1} \rho\left[\sum_{s=1}^{n+k}\left\{\frac{\partial L}{\partial q_s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}\right\}\frac{\partial q_s}{\partial \alpha}\right.$$
$$\left. + \left\{\frac{\partial L}{\partial t} - \frac{d}{dt}\left[L - \sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right]\right\}\frac{\partial t}{\partial \alpha}\right] du$$
$$+ \left[\sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\frac{\partial q_s}{\partial \alpha} + \left(L - \sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\frac{\partial t}{\partial \alpha}\right]_{u_0}^{u_1}.$$

Using (2), (3), and (5), we obtain from this

---

* It is sufficient to require that $\rho = \rho(t, \alpha)$ be a positive integrable function of t for every α.

† The prime denotes differentiation with respect to α.

$$
(14) \quad \begin{aligned} J'(\alpha) = \int_{u_0}^{u_1} \rho \left[\sum_{\nu=1}^{k} \left\{ \sum_{i=1}^{n} a_{\nu i} \frac{\partial q_i}{\partial \alpha} + a_\nu \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha} \right\} \lambda_\nu \right] du \\ + \left[\sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right]_{u_0}^{u_1}. \end{aligned}
$$

Let us now integrate $J'(\alpha)$ around the curve C determined by $u = u_1$:

$$
\int_C J'(\alpha) d\alpha \equiv \int_{\alpha_0}^{\alpha_1} J'(\alpha) d\alpha = J(\alpha_1) - J(\alpha_0) = 0,
$$

since α_0 and α_1 correspond to the same points on the closed curve. Thus we have

$$
(15) \quad \begin{aligned} \int_{u_0}^{\alpha_1} d\alpha \int_{u_0}^{u_1} \rho \left[\sum_{\nu=1}^{k} \left\{ \sum_{i=1}^{n} a_{\nu i} \frac{\partial q_i}{\partial \alpha} + a_\nu \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha} \right\} \lambda_\nu \right] du \\ + \int_{\alpha_0}^{\alpha_1} \left[\sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] d\alpha = \int_{\alpha_0}^{\alpha_1} \sum_{s=1}^{n+k} p_s^0 \frac{\partial q_s^0}{\partial \alpha} d\alpha. \end{aligned}
$$

The integral on the right is taken around the curve C_0, and that on the left around the simple closed circuit C. Since the right side is independent of the circuit C, the expression on the left of (15) is an integral invariant of the Cartan type, as defined in §3. In case the system is holonomic, we get the usual Cartan invariant (see Introduction). Equation (15) includes as a special case the Poincaré invariant

$$
(16) \quad \begin{aligned} \int_{\alpha_0}^{\alpha_1} d\alpha \int_0^{t_1} \left[\sum_{\nu=1}^{k} \left\{ \sum_{i=1}^{n} a_{\nu i} \frac{\partial q_i}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha} \right\} \lambda_\nu \right] dt \\ + \int_{\alpha_0}^{\alpha_1} \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} d\alpha, \end{aligned}
$$

when t_1 is a fixed upper limit. These results may be stated in the following theorem.

THEOREM. *The non-holonomic dynamical system defined by* (1) *and* (2) *admits the relative integral invariant* (15).

5. *A Dynamical Theorem.* We have seen that the equations of motion of a non-holonomic system can be written in the Hamil-

tonian form (4). Suppose that we consider an arbitrary system of differential equations

$$\frac{dq_s}{dt} = Q_s, \qquad \frac{dp_s}{dt} = P_s, \tag{17}$$

where the Q's and P's are any functions of (q_s, p_s, t) with the desired continuity. Let $a_{\nu i}(q, t)$, $a_\nu(q, t)$, $\lambda_\nu(t)$, and $H(q, p, t)$ be given functions, with continuity as required, and suppose that

$$\begin{aligned} &\int_{\alpha_0}^{\alpha_1} d\alpha \int_{u_0}^{u_1} \rho \left[\sum_{\nu=1}^{k} \left\{ \sum_{i=1}^{n} a_{\nu i} \frac{\partial q_i}{\partial \alpha} + a_\nu \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha} \right\} \lambda_\nu \right] du \\ &\quad + \int_{\alpha_0}^{\alpha_1} \left[\sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] d\alpha \end{aligned} \tag{18}$$

is a Cartan integral invariant of the system (17). This means that $u = u_1$ determines a certain simple closed circuit C on the tube of trajectories of (17), and that the integral (18) is independent of u_1. On differentiating with respect to u_1, and dropping the subscript, we have

$$\begin{aligned} &\int_{\alpha_0}^{\alpha_1} \rho \left[\sum_{\nu=1}^{k} \left\{ \sum_{i=1}^{n} a_{\nu i} \frac{\partial q_i}{\partial \alpha} + a_\nu \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha} \right\} \lambda_\nu \right. \\ &\quad \left. + \sum_{s=1}^{n+k} \left\{ \frac{dp_s}{dt} \frac{\partial q_s}{\partial \alpha} + p_s \frac{d}{dt} \frac{\partial q_s}{\partial \alpha} \right\} - \frac{dH}{dt} \frac{\partial t}{\partial \alpha} - H \frac{d}{dt} \frac{\partial t}{\partial \alpha} \right] d\alpha = 0. \end{aligned}$$

Integrating by parts, we find

$$\begin{aligned} &\int_{\alpha_0}^{\alpha_1} \rho \left[\sum_{i=1}^{n} \left\{ \frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} + \sum_{\nu=1}^{k} a_{\nu i} \lambda_\nu \right\} \frac{\partial q_i}{\partial \alpha} \right. \\ &\quad + \sum_{\nu=1}^{k} \left\{ \frac{dp_{n+\nu}}{dt} + \frac{\partial H}{\partial q_{n+\nu}} - \lambda_\nu \right\} \frac{\partial q_{n+\nu}}{\partial \alpha} \\ &\quad + \sum_{s=1}^{n+k} \left\{ -\frac{dq_s}{dt} + \frac{\partial H}{\partial p_s} \right\} \frac{\partial p_s}{\partial \alpha} \\ &\quad \left. + \left\{ \sum_{\nu=1}^{k} a_\nu \lambda_\nu - \frac{dH}{dt} + \frac{\partial H}{\partial t} \right\} \frac{\partial t}{\partial \alpha} \right] d\alpha = 0, \end{aligned}$$

or

$$
(19)\quad
\begin{aligned}
\int_C \rho\Bigg[&\sum_{i=1}^{n}\left\{\frac{dp_i}{dt}+\frac{\partial H}{\partial q_i}+\sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu\right\} dq_i \\
&+\sum_{\nu=1}^{k}\left\{\frac{dp_{n+\nu}}{dt}+\frac{\partial H}{\partial q_{n+\nu}}-\lambda_\nu\right\} dq_{n+\nu} \\
&+\sum_{s=1}^{n+k}\left\{-\frac{dq_s}{dt}+\frac{\partial H}{\partial p_s}\right\} dp_s \\
&+\left\{\sum_{\nu=1}^{k} a_\nu\lambda_\nu-\frac{dH}{dt}+\frac{\partial H}{\partial t}\right\} dt\Bigg]=0.
\end{aligned}
$$

But ρ is a positive function, and, from the way in which it was introduced (see (12b)), it is arbitrary. Hence we may infer that each parenthesis vanishes separately. The resulting equations are precisely (4) and (5), those of a non-holonomic Hamiltonian system. Accordingly we have the following result.

THEOREM. *If the integral* (18) *is a Cartan integral invariant of the system of differential equations* (17), *for a given set of* λ'*s*, $a_{\nu i}$'*s*, *and* a_ν'*s*, *and a given function* H, *then the system* (17) *is of the form* (4), *and consequently there is a non-holonomic dynamical system for which* (17) *are the equations of motion.*

The result is that the non-holonomic system is completely characterized by the integral invariant (18), which may be written more suggestively in the form

$$
\int_C\int_0^t \sum_{\nu=1}^{k}\lambda_\nu\left\{\sum_{i=1}^{n} a_{\nu i}dq_i + a_\nu dt - dq_{n+\nu}\right\} dt + \int_C \sum_{s=1}^{n+k} p_s dq_s - H dt,
$$

it being understood that the upper limit t is a function of α determined by the curve C.

CALIFORNIA INSTITUTE OF TECHNOLOGY