NOTE ON THE ITERATION OF FUNCTIONS OF ONE VARIABLE*

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1. *Introduction.* Let $E(x)$ be a real-valued function of the real variable x for some specified range, and let

$$E_0(x) = x,\ E_1(x) = E(x),\ \cdots,\ E_{n+1}(x) = E(E_n(x)),\ \cdots$$

represent its successive iterations. The interpolation problem of defining $E_n(x)$ for non-integral values of n was discussed some time ago by A. A. Bennett,† who reduced it formally to the solution of the functional equation

$$\psi(x+1) = E(\psi(x)). \tag{1}$$

For if $\psi(x)$ satisfies (1) and if n is any positive integer,

$$\psi(x+n) = E_n(\psi(x)). \tag{2}$$

Hence on writing $\psi^{-1}(x)$ for x, where $\psi^{-1}(x)$ denotes an inverse of the function $\psi(x)$, we obtain the formula

$$E_n(x) = \psi(\psi^{-1}(x) + n), \tag{3}$$

defining $E_n(x)$ for a continuous range of values of n.

In this note, I propose to give an entirely elementary explicit solution to this problem of interpolation for all functions $E(x)$ subject to the following three conditions:‡

(a). *$E(x)$ is a real, continuous, single-valued function of the real variable x in the range $a \leqq x < \infty$.*
(b). *$E(x) > x$ for all $x \geqq a$.*
(c). *$E(x') > E(x)$ if $x' > x \geqq a$.*

We may remark that the functional equation (1) is merely another form of a famous equation studied by Abel,§

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† In two papers in the Annals of Mathematics, (2), vol. 17 (1915–16), pp. 74–75 and pp. 23–60. This second paper contains references to the earlier literature. A. Korkine (Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), pp. 228–242) seems to have been the first to consider this problem.

‡ These conditions are all satisfied by $E(x) = e^x$, the particular case discussed by Bennett in the first paper cited.

§ Works, vol. II, *Posthumous Papers*, 1881, pp. 36–39.

$$(4) \qquad \phi(x) + 1 = \phi(f(x)),$$

as Abel himself showed.* Here $f(x)$ is a given function, and $\phi(x)$ is to be determined. This equation has been extensively investigated of late by modern function-theoretic methods.†

2. *A Simplification.* As a preliminary simplification, we may assume that the constant a in condition (a) is zero, and that $E(0)=1$. For if $E(a)\neq 0$, the function $E'(x)=E^2(x+a)/E^2(a)$ satisfies conditions (a), (b), (c) with $a=0$, while $E'(0)=1$, and $E(x)=\pm E(a)(E'(x-a))^{1/2}$. On the other hand, if $E(a)=0$, then $E_2(a)=E(0)>0$ by (b). Hence $E''(x)=E_2(x+a)/E_2(a)$ will satisfy (a), (b), (c) with $a=0$, $E''(0)=1$. Since $E(x)$ is continuous and monotonic increasing, it has a unique inverse $E_{-1}(x)$. Thus, if $E''(x)$ is given, $E(x)=E_{-1}(E_2(a)E''(x-a))$.

From (b) and (c), it follows that for any positive integer n, $E_n(x')>E_n(x)$ if $x'>x$. Since $E_n(x)$ is furthermore continuous by (a), it has a unique inverse which we shall denote by $E_{-n}(x)$. If we write $y=E_n(x)$, then by (b), $y\geqq E_n(0)$, so that $E_{-n}(x)$ is defined only for $x\geqq E_n(0)$. It is easily verified, however, that for any $x\geqq 0$,

$$(5) \qquad E_n(E_m(x)) = E_{n+m}(x)$$

for all integral values of n and m, positive or negative, for which the functions are defined.

3. *Solution of* (1). We shall next give a solution of the functional equation (1). Let $[x]$ denote as usual the greatest integer in x so that

$$(6) \qquad 0 = E_0(0) \leqq x - [x] < E_1(0) = 1.$$

Then

$$\psi(x) = E_{[x]}(x - [x])$$

is a monotonic increasing continuous solution of (1). For

$$\psi(x+1) = E_{[x+1]}(x + 1 - [x+1]) = E_{[x]+1}(x - [x])$$
$$= E(E_{[x]}(x - [x]) = E(\psi(x)),$$

* Write (1) in the form $x+1=\psi^{-1}(E(\psi(x)))$. Then on substituting $\psi^{-1}(x)$ for x, we obtain $\psi^{-1}(x)+1=\psi^{-1}(E(x))$.

† See, for example, Picard, *Leçons sur Quelques Equations Fonctionnelles*, 1928, Chapter 4. For more recent papers, see the Zentralblatt für Mathematik under the index *Funktionentheorie*: *Iterationen*.

and if $x' \geqq x+1$,

$$\psi(x') = E_{[x']}(x' - [x']) \geqq E_{[x']}(0)$$
$$= E_{[x']-1}(1) \geqq E_{[x]}(1) > E_{[x]}(x - [x]) = \psi(x),$$

while if $x+1 > x' > x$,

$$\psi(x') = E_{[x']}(x' - [x']) = E_{[x]}(x' - [x])$$
$$> E_{[x]}(x - [x]) = \psi(x).$$

The continuity of $\psi(x)$ is obvious if x is not an integer n. Also if $x=n$, $\epsilon>0$, it is clear that $\lim_{\epsilon\to 0}\psi(n+\epsilon)=\psi(n)$. On setting $x=n-\epsilon$, $\epsilon>0$, we have $\lim_{\epsilon\to 0}\psi(n-\epsilon)=\lim_{\epsilon\to 0}E_{n-1}(1-\epsilon)$ $=E_{n-1}(1)=E_n(0)=\psi(n)$.

It follows that $\psi(x)$ has a unique inverse $\psi^{-1}(x)$. To determine it, let x be given, and let the positive integer k be determined by the inequality

$$E_k(0) \leqq x < E_k(1). \tag{7}$$

Then

$$\psi^{-1}(x) = E_{-k}(x) + k.$$

For first of all, $\psi^{-1}(x)$ is defined and continuous for all $x \geqq 0$. Secondly, from (7), $0 \leqq E_{-k}(x) < 1$ so that $k = [\psi^{-1}(x)]$, the greatest integer in $\psi^{-1}(x)$. Therefore

$$\psi(\psi^{-1}(x)) = E_k(\psi^{-1}(x) - k) = E_k(E_{-k}(x)) = E_0(x) = x.$$

Thirdly, since $E_{[x]}(0) \leqq \psi(x) < E_{[x]}(1)$,

$$\psi^{-1}(\psi(x)) = E_{-[x]}(\psi(x) + [x]) + [x]$$
$$= E_{-[x]}(E_{[x]}(x - [x])) + [x] = x.$$

We obtain then, on substituting in (3), the final result of this note:

$$\begin{aligned} E_n(x) &= E_{[n+k+E_{-k}(x)]}(n + k + E_{-k}(x) - [n + k + E_{-k}(x)]) \\ &= E_{[n+k+E_{-k}(x)]}(n + E_{-k}(x) - [n + E_{-k}(x)]). \end{aligned} \tag{8}$$

Here the integer k is determined by the inequality (7) and the formula is valid for all real values of $n \geqq 0$. The equation (5) may now be shown to hold for non-integral values of m and n.

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