NOTE ON A SINGULAR INTEGRAL*

BY E. P. NORTHROP

This note is concerned with the convergence in the mean of order 2, as $m \to \infty$, of the integral

$$(2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(u) K(x - u; m) du.$$

We shall consider necessary and sufficient conditions for the convergence of the integral first to $f(x)$ and secondly to zero. We shall restrict ourselves in this paper to the case where $f(u)$ is of class $L_2(-\infty, +\infty)$, and $K(u; m)$ is of class $L_2(-\infty, +\infty)$ for all values of m. We introduce the following notation:

(a) $f(x) \subset L_p(-\infty, +\infty)$ if and only if $f(x)$ is measurable, and

$$\int_{-\infty}^{+\infty} | f(x) |^p dx < +\infty.$$

(b) Put

$$\|f(x)\|_p \equiv \left[\int_{-\infty}^{+\infty} |f(x)|^p dx \right]^{1/p}.$$

Then the statement "$f_m(x)$ converges to $f(x)$ in the mean of order p as $m \to \infty$" can be written "$\|f_m(x) - f(x)\|_p \to 0$ as $m \to \infty$." We shall also write

$$f(x) = \underset{p}{\text{l.i.m.}}\, f_m(x).$$

(c) Denote by $T[f(x)]$ the Fourier transform of $f(x)$. That is, if $f(x) \subset L_p(-\infty, +\infty)$, $p > 1$, then as $A \to \infty$,

$$T[f(x)] = \underset{p'}{\text{l.i.m.}}\, (2\pi)^{-1/2} \int_{-A}^{A} e^{-ixs} f(s) ds,$$

and $T[f(x)] \subset L_{p'}(-\infty, +\infty)$, where $1/p + 1/p' = 1$. The inverse operator T^{-1} is given by the relation

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$$T^{-1}[f(x)] = \underset{p'}{\text{l.i.m.}}\ (2\pi)^{-1/2}\int_{-A}^{+A} e^{ixs}f(s)ds.$$

Thus, for $1<p\leqq 2$, we have $T^{-1}\{T[f(x)]\} = T\{T^{-1}[f(x)\} = f(x)$.

(d) Put

$$T_m(x;f) \equiv (2\pi)^{-1/2}\int_{-\infty}^{+\infty} f(u)K(x-u;m)du.$$

THEOREM 1. *Let* $f(x) \subset L_2(-\infty, +\infty)$, *and let* $K(x;m) \subset L_2(-\infty, +\infty)$ *for every* m. *Then a set of necessary and sufficient conditions for the convergence of* $\|T_m(x;f)-f(x)\|_2$ *to zero, as* $m\to\infty$, *is*

(i) $|T[K(x;m)]| \leqq M$ *(a constant) for all* m *and almost all* x,

(ii) $$\lim_{m\to\infty}\int_a^b |T[K(x;m)]-1|^2dx = 0,$$

for every finite interval (a,b).

PROOF. Since $f(x)$ and $K(x;m)$ both belong to $L_2(-\infty, +\infty)$, $T_m(x;f)$ is an absolutely convergent integral. In addition,*

$$T_m(x;f) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} e^{ixu}T[f(u)]\cdot T[K(u;m)]du.$$

Here $T[f(u)]\cdot T[K(u;m)] \subset L_1(-\infty, +\infty)$, so that $T_m(x;f)$ is the inverse Fourier transform of this product, and as such, is bounded and continuous. But if the theorem is to have meaning, $T_m(x;f)$ must belong to the class $L_2(-\infty, +\infty)$ for all functions $f(u) \subset L_2(-\infty, +\infty)$, and for every m. This is the case if and only if $T[f(u)]\cdot T[K(u;m)] \subset L_2(-\infty, +\infty)$ for every $f(u) \subset L_2(-\infty, +\infty)$, and for every m. This in turn implies, and is implied by, the existence of a constant M_m, depending only upon m, such that for m fixed, $|T[K(u;m)]| \leqq M_m$ for almost all u. If this is so, then $T[f(x)]\cdot T[K(x;m)]$ is the Fourier transform of $T_m(x;f)$ in $L_2(-\infty, +\infty)$.

Hence to say that $\|T_m(x;f)-f(x)\|_2\to 0$ as $m\to\infty$ is equivalent to saying that $\|T[f(x)]\cdot T[K(x;m)]-T[f(x)]\|_2\to 0$ as $m\to\infty$. But the latter expression is simply the square root of

$$\int_{-\infty}^{+\infty} |T[f(x)]|^2\cdot |T[K(x;m)]-1|^2dx.$$

* See, for example, N. Wiener, Acta Mathematica, vol. 55 (1930), p. 126.

In this integral, $|T[f(x)]|^2$ is an arbitrary positive function of class $L_1(-\infty, +\infty)$, and $|T[K(x; m)]-1|^2$ is a positive function, bounded for every m. By a theorem of H. Lebesgue,* the conditions (i) and (ii) are necessary and sufficient for the convergence of the above integral to zero, as $m\to\infty$. This proves the theorem.

THEOREM 2. *Let condition* (ii) *of Theorem* 1 *be replaced by*

$$\text{(ii)}' \qquad \lim_{m\to\infty}\int_a^b |T[K(x; m)]|^2 dx = 0,$$

for every finite interval (a, b). *Then* (i) *and* (ii′) *are necessary and sufficient in order that* $\|T_m(x;f)-0\|_2=\|T_m(x;f)\|_2\to 0$ *as* $m\to\infty$.

The method of proof is identical with that of Theorem 1. Both Theorem 1 and Theorem 2 can be extended, mutatis mutandis, to cover the case of any number of dimensions.

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A CORRECTION AND AN ADDITION

BY D. N. LEHMER

In an article in this Bulletin (vol. 39 (1933), p. 764), the total number of squares of order four, magic in the rows and columns, is given as 539,136, which arise from 468 "normalized" squares. A second, and after that a third, calculation changes this total to 549,504, which arise from 477 normalized squares. This result is important since it settles definitely in the negative the question as to whether the system of magic squares form a group which is a sub-group of the symmetric group on the number of elements in the square. In fact, the final number contains a factor 53, which is not a factor of the order of the symmetric group on 16 elements. The former erroneous result left this question undecided.

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* Annales de la Faculté des Sciences de l'Université de Toulouse, (3), vol. 1, (1909), p. 52. The results of this paper were previously obtained without the use of this theorem of Lebesgue. M. H. Stone kindly pointed out to the author, however, that the work could be considerably shortened by using it.