WARING'S PROBLEM FOR NINTH POWERS*

BY L. E. DICKSON

1. *Introduction.* In a previous paper in this Bulletin (vol. 39 (1933), p. 701), I gave a method to obtain universal Waring theorems by supplementing the asymptotic results obtained by the analytic theory of Hardy and Littlewood. I quoted results obtained from tables of all minimum decompositions into powers. Later I discovered an ideal method of making such a table, which is now algebraic rather than numerical. Quite recently, I found that we can greatly shorten the work and the table itself if we do not require that our decompositions be minimal. We may discard more than half the linear functions necessary for a minimal table.

The conclusion is that every positive integer is a sum of 981 integral ninth powers $\geqq 0$. This is close to the asymptotic result 949 by Hardy and Littlewood.

2. *Notation and Equations.* Write

$$a=2^9=512, \quad b=3^9, \quad c=4^9, \quad d=5^9, \quad f=6^9.$$

Then

$$(1)\quad b = 227 + 38a, \qquad c = 121 + 12a + 13b,$$

$$(2)\quad d = 19 + 6b + 7c, \qquad f = 321 + 20a + 2b + c + 5d.$$

We employ 44 linear equations which follow algebraically from (1), (2), and $a=512$. No. 5 in §3 is

$$(3)\qquad 39a + 5b + 7c - d = 266,$$

which follows from the left-hand equations (1), (2). We obtain our further equations Nos. 3–7 involving $-d$ from (3) by additions or subtractions of the pair (1). Of the equations Nos. 8–13 involving $-2d$, No. 9 is the double of (2_1), while the others follow from it by additions and subtractions of (1). Again, (2_2) implies No. 28, from which we obtain Nos. 14–44 by additions and subtractions of (1) and (2_1).

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3. *List of* 44 *Linear Equations.*

1. $39a-b=285$, wt. 38 2. $13a+13b-c=391$, wt. 25

No.	a	b	c	d	f	r	wt.
3.	37	45	4	−1		130	85
4.	13	19	6	−1		372	37
5.	39	5	7	−1		266	50
6.	−26	20	6	−1		87	−1
7.	27	−8	8	−1		387	26
8.	13	25	13	−2		353	49
9.	1	12	14	−2		474	25
10.	−1	52	11	−2		338	60
11.	−13	39	12	−2		459	36
12.	−26	26	13	−2		68	11
13.	−12	−1	15	−2		83	0
14.	35	5	38	0	−1	111	77
15.	21	26	29	1	−1	115	76
16.	9	13	30	1	−1	236	52
17.	7	47	20	2	−1	119	75
18.	21	20	22	2	−1	134	64
19.	9	7	23	2	−1	255	40
20.	60	13	15	3	−1	438	90
21.	21	14	15	3	−1	153	52
22.	9	1	16	3	−1	274	28
23.	46	34	6	4	−1	442	89
24.	7	35	6	4	−1	157	51
25.	60	7	8	4	−1	457	78
26.	21	8	8	4	−1	172	40
27.	33	15	0	5	−1	70	52
28.	21	2	1	5	−1	191	28
29.	−3	6	38	0	−1	338	40
30.	−3	0	31	1	−1	357	28
31.	−5	16	0	5	−1	297	15
32.	−17	9	8	4	−1	399	3
33.	−21	95	9	3	−1	108	85
34.	−33	82	10	3	−1	229	61
35.	−33	76	3	4	−1	248	49
36.	−30	−4	9	4	−1	8	−22
37.	9	−5	9	4	−1	293	16
38.	−3	−6	24	2	−1	376	16
39.	23	−8	39	0	−1	232	53
40.	47	−12	2	5	−1	85	41
41.	−3	−12	17	3	−1	395	4
42.	46	28	−1	5	−1	461	77
43.	−15	−1	46	−1	−1	440	28
44.	−15	5	53	−2	−1	421	40

4. *Table of Decompositions of Integers* $N>2d+f$. We may express N as $r+Aa+Bb+Cc+2d+f$, where $0\leqq r<a$, $r+Aa<b$, etc. Hence $A\leqq 38$, $B\leqq 13$, $C\leqq 7$ by (1)–(2).

In tablette I, $A=0$, second line, 27 is the number of the function in §3, and 70 is its r. To explain the remaining entry 100, we add 1 less the tabular difference 119–70 to the weight 52 (sum of coefficients) of function No. 27. Evidently the sum of No. 27 and $2d+f$ is a decomposition of $70+2d+f$. Hence for $r=70-118$, $r+2d+f$ has a decomposition into 100 ninth powers. The largest number in the second column gives the maximum 106 printed below it, whence 106 powers suffice for all integers $k+2d+f$, $0\leqq k\leqq 511$.

Thus in tablette I, $A=15$, the sum of No. 27 and $15a+2d+f$ is a decomposition of $70+15a+2d+f$. Here we may use only functions in which the coefficient of a is $\geqq -15$, while (as for tablette $A=0$) the coefficients of b and c are positive.

I. $B=C=0$

$A=0$

0	69	0
27	100	70
17	85	119
3	88	130
18	82	134
21	89	153
28	102	191
5	57	266
22	106	274
8	67	353
4	102	372
20	93	438
23	103	442
25	94	457
9	62	474
	Max 106	

$A=1$–14

22	91	274
10	74	338
8–, $A=0$		
	Max 103	

$A=15$–20

0	69	0
27	92	70
14	84	111
17–21, $A=0$		
28	72	191
16	81	236
5	57	266
22	91	274
29	54	338
8	67	353
4	85	372
44	77	421
11	50	459
9	62	474
	Max 92	

$A=21$–25

0	69	0
27	89	70
33	87	108
14	84	111
17–21, $A=0$		
28	72	191
16	89	236
22	50	274
31	55	297
29	54	338
8	67	353
4–, $A=15$		
	Max 89	

$A=26$–32

0	67	0
12	29	68
6	64	87
21	55	153
24	65	157
26	58	172
28	72	191
16	70	236
19	58	255
22–8, $A=21$		
4	63	372
32	62	399
11	50	459
9	62	474
	Max 72	

$A=33$–38

0–26, $A=26$		
28	65	191
34	67	229
16	63	236
35	55	248
19	58	255
22–8, $A=21$		
4–, $A=26$		
	Max 67	

II. $B=1$, $C=0$

$A=15$

0	69	0
27	64	70
13	69	83
21	70	153
26	58	172
28–5, I, $A=15$		
22	50	274
31	70	297
8	67	353
4	85	372
44–, I, $A=15$		

Max 85

III. $B=6$, $C=0$

A $=0$–2

22	46	274
37	75	293
8		353

Max 103

$A=3$–14

0	69	0
27	92	70
14	84	111
17–21, $B=A=0$		
28	72	191
16	81	236
5	57	266
22	38	274
1	90	285
29	54	338
8	67	353
4	40	372
38	96	376
25	94	457
9	62	474

Max 96

$A=15$–32

0–26, II		
28	72	191
16	70	236
19	50	255
5	57	266
22–8, II		
4	40	372
38	60	376
44	58	421
43	46	440
11	50	459
9	62	474

Max 72

IV. $B=8$, $C=0$, $A=0$

0	69	0
27	92	70
14	84	111
17–21, I		
28	72	191
16	89	236
22	38	274
1	45	285
37	75	293
8	67	353
4	51	372
7	95	387
25	94	457
9	62	474

Max 95

V. $B=12$, $C=0$

$A=3$

0	69	0
27	66	70
40	85	85
3	88	130
18	82	134
21	70	153
26	58	172
28	68	191
39	56	232
16	70	236
19	50	255
5	57	226
22–4, IV		
7	33	387
41	82	395
9	62	474

Max 88

$A=12$

0	69	0
27	64	70
13	69	83
21–, $A=3$		

Max 82

$A=13$

0–13, $A=12$		
21–5, $A=3$		
22, 1, IV		
37	19	293
31	55	297
29	54	338
8	67	353
4	59	372
38	67	395
11	50	459
9	62	474

Max 70

$A=33$

0	7	0
36	37	8
12–26, I, $A=26$		
28	65	191
34	63	229
39	56	232
16–19, I, $A=33$		
22–29, I, $A=21$		
8	52	353
30	42	357
4–, I, $A=26$		

Max 65

VI. $C=1$, $B=0$

$A=1$

0	69	0
27	92	70
14	84	111
17–21, I, $A=0$		
28	91	191
19	58	255
22	91	274
10	74	338
8	67	353
4	55	372
2	90	391
25, 9, I, $A=0$		

Max 94

VII. $C=1$, $B=8$				$A=13$	
	$A=0$		0	69	0
0–4, IV			27	64	70
7	29	387	13	69	83
2	90	391	21	89	153
25	81	457	28–4, IV		
42	89	461	7, 2, $A=0$		
9	62	474	25	79	457
	Max 92		11	50	459
			9	62	474
				Max 90	

5. *Conclusions from the Table.* Let G denote the greatest weight (second column) in a tablette with fixed A, B, C. By I ($B=C=0$), we have

(4)

A	0	1–14	15–20	21–25	26–32	33–38
G	106	103	92	89	72	67
$A+G$	106	117	112	114	104	105

For $A+G$ the largest A was used. Including 3 (from $2d+f$), we see that $3+117=120$ ninth powers suffice for every A if $B=C=0$. Hence for $C=0$, $B=0-5$, 125 powers suffice.

Let $B=6$, $C=0$. By III, $A+G=2+103$ if $A=0-2$, $A+G=14+96=110$ if $A=3$–14, $A+G=32+72=104$ if $A=15-32$. Also $A+G=105$ if $A=33$–38 by (4). Hence $A+G\leqq 110$ for all A. Thus $3+6+110=119$ powers suffice for all A if $B=6$, $C=0$.

Let $B=8$, $C=0$. By IV, $A+G=14+95=109$ if $A=0$–14. By $B=6$, $A+G\leqq 105$ if $A\geqq 15$. Hence $3+8+109=120$ powers suffice for all A.

Let $B=12$, $C=0$. By IV, V, we have

(5)

A	0–2	3–11	12	13–32	33–38
G	95	88	82	70	65
$A+G$	97	99	94	102	103

Thus $3+12+103=118$ powers suffice for all A. Since $B\leqq 13$, our results may be combined into the following lemma.

LEMMA 1. *If* $C=0$, 125 *ninth powers suffice for all* A, B, *while* 124 *suffice except when* $A=14$, $B=5$, $r=456$.

Let $C=1$, $B=0$. By VI, $A+G=108$ if $A=1$–14. By (4), $A+G\leqq 114$ unless $A=1$–14. Hence $3+1+114=118$ powers suffice for all A if $C=1$, $B=0$.

Let $C=B=1$. By II, $A+G=25+85=110$ if $A=15$–25. By the preceding, $A+G=108$ if $A=1$–14. By (4), $A+G=106$ if $A=0$ and $A+G\leqq 105$ if $A=26$–38. Hence $3+2+110=115$ powers suffice for all A if $C=B=1$. Thus 119 powers suffice if $C=1$, $B=1$–5.

Let $C=1$, $B=6$. If $A=15$–25, $A+G=25+72=97$ by III. By the preceding ($C=B=1$), $A+G\leqq 108$ if $A=0$–14, 26–38. Hence $3+7+108=118$ powers suffice for all A. Thus 119 powers suffice if $C=1$, $B=0$–7.

LEMMA 2. 125 *ninth powers suffice if* $C=1$–7, $B\leqq 7$.

Let $C=1$, $B=8$. By VII, $A+G=104$ if $A=0$–12, and if $A=13$, 14. By III, $A+G=104$ if $A=15$–32. By (4), $A+G=105$ if $A=33$–38. Hence $3+9+105=117$ powers suffice for all A. But $C<7$ if $B>6$ by (2_1).

LEMMA 3. 125 *ninth powers suffice if* $C\geqq 1$, $B=8$–11.

By (5), 120 powers suffice if $C=1$, $B=12$, 13. Since $C<7$ if $B>6$, 125 powers suffice if $C\geqq 1$, $B\geqq 12$.

Our results together show that 125 powers suffice for all A, B, C yielding integers in our interval of length d. This may be expressed as follows.

THEOREM 1. *All integers from* $2d+f$ *to* $3d+f$ *are sums of* 125 *ninth powers.*

Our results show also that 124 suffice for all A, B if $C=1$–5. Adding d we see that 125 suffice from $c+3d+f$ to $6c+3d+f$. To the exceptional number in Lemma 1 we add d and get $N=456+14a+5b+3d+f$. Eliminate d by inserting the triple of its value (2). Thus $N=1+15a+23b+21c+f$, whose weight is 61. This proves the following extension of Theorem 1.

THEOREM 2. *All integers from* $h=2d+f$ *to* $6c+3d+f$ *are sums of* 125 *ninth powers.*

6. *The Universal Theorem.* We add d and f each three times, 7^9 twice, and n^9 ($n=8$–13, 15) each once, as in Theorem 10, this Bulletin (vol. 39 (1930), p. 710), and find that 140 powers suffice from h to $L_0=58221534000$. By Theorem 12, ibid., page 711, with $t=841$, we find that 981 powers suffice from h to L_t, where log log $L_t=43.356$. By R. D. James' recent work for odd powers, every integer $>C$ is a sum of 981 ninth powers if log log $C=43.198$. Since $L_t>C$, we have the following result.

THEOREM 3. *All integers* $\geqq 2d+f$ *are sums of* 981 *ninth powers.*

By (1_1) the integers $\geqq 0$ and $<b$ are $x+ya$ ($x\leqq 511$, $y\leqq 37$) and $z+38a$ ($z\leqq 226$), and hence are sums of 548 powers. Adding b thirteen times, we see that 561 suffice to $14b$ and hence beyond c. Adding c seven times, we see that 568 suffice to $8c>d$. Adding d five times, we see that 573 suffice to $6d>f$. Hence 575 suffice to $2d+f$.

THEOREM 4. *Every positive integer is a sum of* 981 *ninth powers.*

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A NOTE

BY R. L. PEEK, JR.

M. Maurice Fréchet, of the University of Paris, has informed me that Cantelli published the following inequality in the Bollettino dell' Associazione degli Attuari Italiani (Milan, 1910):

$$P_{|X-Y|} \geqq \epsilon \geqq \frac{M_{2r} - M_r^2}{(\epsilon^r - M_r)^2 + M_{2r} - M_r^{\,r}},$$

where M_r is the mean of $|X-Y|^r$. As Fréchet pointed out in his letter to me, this inequality includes as a particular case ($Y=\overline{X}$, $\epsilon=t\sigma$, $r=1$) the inequality (2) given in my paper, *Some new theorems on limits of variation*, published in this Bulletin, December, 1933.

The journal in which Cantelli's paper appeared is not, so far as I have been able to ascertain, available in New York City.

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