AN INVOLUTORIAL LINE TRANSFORMATION DETERMINED BY A BILINEAR CONGRUENCE OF TWISTED ELLIPTIC QUARTIC CURVES*

BY VIRGIL SNYDER AND J. M. CLARKSON

1. *Introduction.* Let there be given two elliptic space quartic curves α, β, bases, respectively, of the two pencils of quadrics $H_1-\alpha H_2=0$, and $K_1-\beta K_2=0$. The curve $C_4(\alpha,\beta)$ of intersection of a quadric of one pencil with one of the other meets each of α,β in 8 points. As the parameters α, β take on all values independently, $C_4(\alpha,\beta)$ describes a system of ∞^2 (a congruence of) elliptic space quartics. Through an arbitrary point (u) passes just one $C_4(\alpha,\beta)$, namely that for which $\alpha=H_1(u)/H_2(u)$ and $\beta=K_1(u)/K_2(u)$.

A quadric of the system

$$(1)\qquad (H_1-\alpha H_2)-\rho(K_1-\beta K_2)=0$$

is determined by three independent linear relations among α, β, ρ, hence by any three points of space. If these three points be chosen on a straight line t, then the quadric of (1) determined by the three points contains t as a generator. Thus t is a bisecant of every elliptic quartic lying on the quadric. But the values of α, β so determined fix a $C_4(\alpha,\beta)$ of the congruence and it lies on the quadric of (1). Hence an arbitrary line t of space is bisecant to just one $C_4(\alpha,\beta)$.

Now, let $\gamma\equiv\sum_{i=1}^4 c_iz_i=0$ be an arbitrary fixed plane. Any line t meets γ in a point P. The quadric $Q(t)$ of (1) which contains t as a generator has another generator t' also passing through P, and t' is likewise bisecant to the $C_4(\alpha,\beta)$ determined by t. The line transformation $t\sim t'$ is involutorial and birational. It is the purpose of this paper to study this involution I.†

* Presented to the Society, March 30, 1934.

† A brief synthetic outline, mostly without proofs, of parts of this paper is given by J. de Vries: *On an involution among the rays of space, which is determined by a bilinear congruence of twisted elliptical quartics*, Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 22 (1919), pp. 493–496.

2. *The Order of the Transformation.* Let the Plücker coordinates of t be y_i and those of t' be x_i, $(i=1, 2, \cdots, 6)$. The point P in which t meets γ has coordinates which are linear in y_i, and any other two points A, B on t have coordinates each linear in y_i. The quadric $Q(t)$ of the system (1) containing t as a generator is

$$\begin{aligned} K_2^6(y)\,[H_2^6(y)H_1(z) - H_1^6(y)H_2(z)] \\ + [H_2^6(y)K_2^6(y)K_1(z) - K_1^6(y)K_2(z)] = 0, \end{aligned} \tag{2}$$

where

$$H_1^6(y) = \begin{vmatrix} H_1(A, B) & H_1(A) & H_1(B) \\ K_1(A, B) & K_1(A) & K_1(B) \\ K_2(A, B) & K_2(A) & K_2(B) \end{vmatrix},$$

$$H_2^6(y) = \begin{vmatrix} H_2(A, B) & H_2(A) & H_2(B) \\ K_1(A, B) & K_1(A) & K_1(B) \\ K_2(A, B) & K_2(A) & K_2(B) \end{vmatrix},$$

$$K_1^6(y) = \begin{vmatrix} K_1(A, B) & K_1(A) & K_1(B) \\ H_1(A, B) & H_1(A) & H_1(B) \\ H_2(A, B) & H_2(A) & H_2(B) \end{vmatrix},$$

$$K_2^6(y) = \begin{vmatrix} K_2(A, B) & K_2(A) & K_2(B) \\ H_1(A, B) & H_1(A) & H_1(B) \\ H_2(A, B) & H_2(A) & H_2(B) \end{vmatrix}.$$

The parameters λ, μ of the two reguli on $Q(t)$ are each of degree 12 in y_i. The Plücker coordinates of a generator of the λ-regulus are of degree 2 in λ and those of a generator of the μ-regulus are of degree 2 in μ. If now we consider t' as being of the λ-regulus and t of the μ-regulus, we have

$$\xi x_i = \phi_i(y), \qquad (i = 1, 2, \cdots, 6), \tag{3}$$

where the ϕ_i are functions of degree 24 in y_j, and ξ is a constant. Thus the line transformation $(3) \equiv t \sim t'$ is of order 24.

3. *The Singular Lines of the Transformation.* Suppose the line

a to be bisecant to the fixed quartic $\alpha \equiv H_1 = H_2 = 0$. There is one quadric of the pencil $H_1 - \alpha H_2 = 0$ which contains a as a generator. Through each point of a passes just one quadric of the second pencil $K_1 - \beta K_2 = 0$, and hence a is bisecant to ∞^1 $C_4(\alpha, \beta)$ of the congruence. However, since the conjugate a' of a in I must pass through the point where a meets λ, a' is uniquely determined and is bisecant to only one of the ∞^1 $C_4(\alpha, \beta)$ met twice by a. Thus a is not singular in I. Also, the lines b bisecant to the fixed quartic $\beta \equiv K_1 = K_2 = 0$ are not singular.

Can there exist a line s not bisecant to either fixed curve α, β and yet bisecant to ∞^1 $C_4(\alpha, \beta)$ of the congruence?

Let (u) be a fixed point of space. It determines the quadric $H(u) \equiv H_2(u)H_1(z) - H_1(u)H_2(z) = 0$ of the first pencil and $K(u) \equiv K_2(u)K_1(z) - K_1(u)K_2(z) = 0$ of the second. $H(u)$ and $K(u)$ meet in $C_4(u)$. Let (v) be any other point on $C_4(u)$. Then

$$(4) \qquad \begin{cases} H_2(u)H_1(v) - H_1(u)H_2(v) = 0, \\ K_2(u)K_1(v) - K_1(u)K_2(v) = 0. \end{cases}$$

Let $\lambda u + \bar{\mu} v$ be a fixed point on the line $s \equiv (u)(v)$. The quadric $H(\bar{\lambda} u + \bar{\mu} v)$ determined by it meets s in another point $\lambda u + \mu v$, where

$$(5) \qquad \begin{cases} \lambda = \bar{\mu}[H_2(v)H_1(u, v) - H_1(v)H_2(u, v)], \\ \mu = \bar{\lambda}[H_2(u)H_1(v, u) - H_1(u)H_2(v, u)]. \end{cases}$$

If $K(\bar{\lambda} u + \bar{\mu} v)$ also passes through $\lambda u + \mu v$, we have

$$(6) \qquad \frac{H_2(v)H_1(u, v) - H_1(v)H_2(u, v)}{H_2(u)H_1(v, u) - H_1(u)H_2(v, u)} = \frac{K_2(v)K_1(u, v) - K_1(v)K_2(u, v)}{K_2(u)K_1(v, u) - K_1(u)K_2(v, u)},$$

which is independent of the ratio $\bar{\lambda}/\bar{\mu}$. Thus if (6) and the preceding conditions are satisfied, every $C_4(\alpha, \beta)$ of an entire pencil has s as a bisecant. Hence s is fundamental in I.

From (4), we have

$$(7) \qquad \begin{cases} H_1(u)/H_2(u) = H_1(v)/H_2(v) = p, \\ K_1(u)/K_2(u) = K_1(v)/K_2(v) = q. \end{cases}$$

Substituting (7) in (6) we have

$$(8) \qquad H_2(v)/H_2(u) = K_2(v)/K_2(u),$$

provided $H_1(u, v) \neq pH_2(u, v)$ and $K_1(u, v) \neq qK_2(u, v)$. Thus, from (8) and (7), we have

$$(9) \qquad \frac{H_1(v)}{H_1(u)} = \frac{H_2(v)}{H_2(u)} = \frac{K_1(v)}{K_1(u)} = \frac{K_2(v)}{K_2(u)},$$

or, if $H_1(u, v) = pH_2(u, v)$ and $K_1(u, v) = qK_2(u, v)$,

$$(10) \qquad \begin{cases} \dfrac{H_1(u)}{H_2(u)} = \dfrac{H_1(v)}{H_2(v)} = \dfrac{H_1(u, v)}{H_2(u, v)}, \\[2ex] \dfrac{K_1(u)}{K_2(u)} = \dfrac{K_1(v)}{K_2(v)} = \dfrac{K_1(u, v)}{K_2(u, v)}. \end{cases}$$

But the first line of (10) states that the entire line s lies on $H(u)$ and $H(v)$, while the second line makes a similar statement concerning $K(u)$ and $K(v)$. Thus every $C_4(\alpha, \beta)$ of the pencil $\bar{\lambda}/\bar{\mu}$ has s as a component, and $\lambda = \mu = 0$. Hence the involution on s is established by the equations (9).

Now let the point (u) be an arbitrary point of space. If (9) are satisfied, then (v) must be one of the base points of a net of quadrics, another of which base points is (u). Hence, through an arbitrary point of space pass 7 fundamental lines s of I.

Since λ/μ depends linearly on $\bar{\lambda}/\bar{\mu}$, the two pencils $C_4(\lambda u + \mu v)$ and $C_4(\bar{\lambda} u + \bar{\mu} v)$ of quartics of the congruence are projective. These curves generate a quartic surface F_4 which contains the line s. A plane π through s meets each C_4 of the two pencils in two other points, each of which determines the other uniquely. This involution in π is rational and hence must be central. The residual intersection of F_4 by π is a cubic π_3 generated by the pairs of points of the involution. The lines t' in π pass through a point P on π_3. As π turns about s, P describes a curve.

Among the quadrics of the pencil

$$(11) \qquad (H_1 - \lambda H_2) - \rho(K_1 - \mu K_2) = 0, \qquad (\mu \text{ projective with } \lambda),$$

one contains s. This quadric meets F_4 in s and a residual curve C_3, which is the locus of P. The curve C_3 is a space cubic meeting s twice. C_3 and s form the base of a pencil of quadrics each of which meets F_4 again in a $C_4(\alpha, \beta)$ of the original congruence.

From any point on s, say Q, in any plane π through s passes one line of the pencil through P. Thus Q is the vertex of a cubic cone with s as double generator, each generator of which meets some C_4 of the pencil twice. Hence s is singular in I. Also, since an arbitrary plane ϕ meets s in some point Q, in ϕ there lie 3 bisecants of curves C_4 of the pencil. Thus, *the fundamental lines s form a congruence* (7, 3).

Through an arbitrary point P of the plane γ passes one $C_4(\alpha, \beta)$. The bisecants of this $C_4(P)$ through P generate a cubic cone every generator s^* of which is the conjugate in I of any one of them. These lines s^* are therefore fundamental of the second kind and are also on the locus of invariant lines of I. They form a complex whose order is discussed in §5.

Any line t_γ in the plane γ is bisecant to one $C_4(\alpha, \beta)$. The bisecants of this $C_4(t_\gamma)$ which meet t_γ belong to one regulus of the quadric $Q(t_\gamma)$ of (1) and t_γ belongs to the other regulus of $Q(t_\gamma)$. Thus the conjugate of t_γ in I is this quadric regulus, plus the two cubic cones of the complex (s^*) whose vertices are the points where t_γ meets $C_4(t_\gamma)$. *The plane field* $[\gamma]$ *of lines is fundamental.*

4. *The Invariant Lines of the Transformation.* The invariant lines of (3) form a complex whose order is discussed in §5.

5. *Conjugates in I of a Pencil, a Bundle, and a Plane Field of Lines.* Given a pencil of lines (T, τ). Each line t of the pencil is bisecant to one $C_4(\alpha, \beta)$. We shall define the order of the surface ψ generated by $C_4(t)$ as t describes (T, τ).

Let P be any point on the fixed curve α. Through P pass ∞^1 $C_4(\alpha, \beta)$, the intersections of the quadric $K(P)$ and the pencil $H_1 - \alpha H_2 = 0$. The quadric $K(P)$ meets τ in a conic. The pencil $H_1 - \alpha H_2 = 0$ meets this conic in the groups of an I_4, the points of each group lying on a C_4 of the system. Let A be any point on the fixed conic $K(P)$, τ. The conic $H(A)$, τ meets the fixed conic in A and three other points A'. The line TA meets the fixed conic in one other point B. How often does B coincide with one of the points A'?

Let $f=0$ be the conic $K(P)$, τ, and $\phi - \lambda\phi' = 0$ be a conic of the pencil. Through the points of intersection of f and $\phi - \lambda\phi'$ pass a third conic through T:

$$f(\phi_0 - \lambda\phi_0') - f_0(\phi - \lambda\phi') = 0. \tag{12}$$

When (12) is composite one component passes through T and meets a C_4 of the ∞^1 C_4 through P. The discriminant of (12) is cubic in λ, and hence there are three lines of (T, τ) each a bisecant to one C_4 of the pencil through P. On the surface ψ the fixed quartic α is triple. In like manner β is also triple.

The quadric $K(P)$ meets ψ in the three generating C_4 through P and in the curve β counted three times. Thus the order of the complete intersection of $K(P)$ and ψ is 24. Hence ψ is of order 12.

Each line of (T, τ) meets its associated C_4 in two points. There is one $C_4(\alpha, \beta)$ passing through T. This $C_4(T)$ meets τ in three other points each of which makes with T a corresponding pair. Hence T is triple on the locus of associated pairs, and this locus is therefore a plane quintic $\tau_5\colon T^3$. The quintic τ_5 passes through the four points α, τ and the four points β, τ.

If (L, λ) is any other pencil of lines and λ_5 the corresponding quintic curve, then λ_5 meets ψ_{12} in 60 points, 12 of which are on α and 12 on β. The other 36 points must be arranged in 18 pairs. The pencil (L, λ) then contains 18 bisecants of the C_4 which generate ψ_{12}; hence *the bisecants of the* ∞^1 C_4 *having each one bisecant belonging to a given pencil* (T, τ) *form a line complex of order* 18.

The curve τ_5 discussed immediately above meets the line τ, γ in 5 points P, and hence the pencil (T, τ) contains 5 lines s^* (see §3). Thus the complex (s^*) is of order 5. It is the locus of invariant lines of the transformation (3).

We shall now determine the order of the ruled surface ϕ, conjugate under I of the pencil (T, τ). The lines t of (T, τ) meet γ in the points of τ, γ. The curve τ_5, locus of the pairs of points in which t meets its associated $C_4(\alpha, \beta)$, meets τ, γ in 5 points P_0, the conjugate of t through each P_0 being the generators of an elliptic cubic cone, vertex at P_0. Through each point of τ, γ other than P_0 passes only one generator of ϕ, and hence τ, γ is a simple directrix of ϕ. Thus the order of ϕ is one more than the number of lines in which an arbitrary plane through τ, γ meets ϕ (other than τ, γ itself). When t is given a $C_4(\alpha, \beta)$ is fixed. This $C_4(t)$ meets τ in 4 points, two of which are on t, the other two on a line l meeting t. As t describes (T, τ), the line l envelops a conic and the point t, l traces a cubic curve in τ. This cubic meets τ, γ in three points Q_0 at each of which $l \equiv t'$. Thus

τ meets ϕ in five lines TP_0, three lines TQ_0, and in the line τ, γ. Hence ϕ is of order 9 and *the conjugate under I of an arbitrary plane pencil (T, τ) is a composite ruled surface of order* 24 *consisting of a rational ruled surface of order* 9 *and five elliptic cubic cones.*

A bundle of lines $[M]$ with vertex M is transformed by I into a congruence. An arbitrary line t of $[M]$ is bisecant to just one $C_4(\alpha, \beta)$; through an arbitrary point N pass two bisecants u_1, u_2 of this $C_4(t)$. The line t meets γ in a point P and u_1, u_2 meet γ in two points Q_1, Q_2. Then Q_1, Q_2 correspond to P. To each point Q correspond two points P_1, P_2. Thus there is set up in γ a (2, 2) correspondence. Whenever it happens that P coincides with either Q_1 or Q_2, then the conjugate of t in I is the line u_1 or u_2. Since N was chosen arbitrarily, there can be in general only a finite number of such coincidences in γ. Now, as P describes a line in γ, the line t describes a pencil of $|M|$ and we have seen (§5) that the bisecants of the $C_4(\alpha, \beta)$ to which the lines of a pencil are bisecants form a complex of order 18. Hence Q_1 and Q_2 describe a curve of order 18 in γ. The number ξ of coincidences in an (α_1, α_2) correspondence in a plane is given by

$$\xi = \alpha_1 + \alpha_2 + \beta - \eta - \zeta, \tag{13}$$

where β is the number of points Q on an arbitrary line whose corresponding points P lie on another arbitrary line, η the order of the curve each point of which is a coincidence, and ζ the class of the curve of coincidences.† Thus in our case we have

$$\xi = 2 + 2 + 18 - 0 - 0 = 22. \tag{13'}$$

Hence the order of the congruence which is the conjugate under I of $[M]$ is 22.

The curves $C_4(\alpha, \beta)$ having lines of $[M]$ as bisecants generate a surface of order 5 (see §5, fifth paragraph). Hence in γ there is a curve γ_5 each point of which is the vertex of a cubic cone of singular lines s^* (§3). These lines s^* are also invariant under I and hence through M pass all of the generators of a quintic cone each of which is invariant. Thus M is a singular point of fifth order for the conjugate congruence.

† H. G. Zeuthen, *Lehrbuch der Abzählenden Geometrie*, pp. 271–274.

Let μ be an arbitrary plane of space, ν the plane through M and γ, μ. To each line t of $[M]$ correspond six bisecants of $C_4(t)$ lying in μ. Let $Q_1, Q_2, \cdots, Q_6$ be their points of intersection with γ, μ, and let P be the point where t meets γ. We say that $Q_1, \cdots, Q_6$ correspond to P. The line complex of order 18 corresponding to the pencil (Q, μ) has 18 lines in the pencil (M, ν); thus to each Q correspond 18 points P. All of the points Q lie on γ, μ and hence as P describes a straight line in γ there will be in general no points Q on an arbitrary line in γ. Formula (13) then becomes

$$(13'') \qquad \xi = 6 + 18 + 0 - 0 - 0 = 24.$$

The class of the congruence conjugate to $[M]$ is 24.

The transformation (3) is involutorial, and so the order of the congruence conjugate to an arbitrary plane field $[\mu]$ is 24, the number of lines common to the conjugate of an arbitrary bundle $[M]$ and to the plane field $[\mu]$. The class is found as follows. The only lines t in μ whose conjugates t' can lie in an arbitrary plane ν must pass through the point $O \equiv \gamma;\ \mu,\ \nu$ and the lines t' must also pass through O. The ruled surface ϕ_{24} conjugate to the pencil (O, μ) breaks up into the pencil (O, μ), the cubic cone that projects $C_4(O)$ from O counted three times (once for each of the three generators belonging to (O, μ)) and a cone of order 14. Hence in ν lie 23 lines t' conjugate to lines t in μ. *Therefore the conjugate under I of a plane field $[\mu]$ of lines is a congruence* (24, 23).

Cornell University