matrices are similar if and only if they belong to the same class. Let

$$f(x) = x^n + k_1x^{n-1} + \cdots + k_n,$$

where the k's are rational integers, $k_n \neq 0$, and $f(x) = 0$ has no multiple roots. If A is a matric root of $f(x) = 0$ and is non-derogatory, that is, is not a root of an equation, with rational coefficients, of lower degree, the same is true of every matrix similar to A. It is known that there is a one-to-one correspondence between the classes of ideals in a domain of integrity in a certain commutative semi-simple algebra and the classes of non-derogatory matrices which are roots of $f(x) = 0$.* We have therefore, by Theorem 1, the following result.

THEOREM 2. *The number of classes of non-derogatory similar matrices which are roots of* $f(x) = 0$ *is finite.*

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KASNER'S CONVEX CURVES†

BY MORRIS HALPERIN

1. *Preliminary Discussion.* A Kasner convex curve is the limit of a sequence of simple, closed, convex polygons, $P_0, \cdots, P_n, \cdots$, each of which has a finite number of sides and is obtained from the preceding one by measuring off the rth part of the length of each side from both its ends and cutting off the corners. The number r is restricted to the inequality $0 < r < 1/2$. To obtain an analytic definition for the curve, we proceed as follows. We note that the centroid of the vertices of P_0 is also the centroid of the vertices of every P_n. Hence G is interior to every P_n. Let $z_n(t)$ be the intersection of a ray from G of inclination t with the polygon P_n. The sequence of functions $\{z_n(t)\}$ will be found to converge uniformly to a function $z(t)$.

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* Latimer and MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Annals of Mathematics, (2), vol. 34 (1933), pp. 313–316.

† Presented to the Society, February 25, 1933. Another paper will follow in which additional properties of these curves will be discussed; particularly their second derivatives, their non-analytic character, and their areas. See this Bulletin, Abstract 39-3-68.

The curve defined by this function for $0 \leqq t \leqq 2\pi$, is found to be simple, closed, continuous, and convex. We shall call this curve a *Kasner convex curve*. The symbol K will be used to represent this curve.

By the *M-points of* P_n, we shall mean the midpoints of the sides of P_n. It is easily verified that:

1. Every M-point of P_n is an M-point of P_{n+1}, and hence of P_{n+p} for every positive integral value of p. Consequently, every M-point of every P_n is a point of K.

2. If Q is a non-M-point of P_n, a number p exists such that Q is exterior to P_{n+p}. Since every point of K is on or within every P_{n+p}, it follows that Q is not on K.

3. The maximum distance between two successive M-points of P_n decreases to zero as n increases to infinity. Hence the M-points of all the P_n form a set dense on K.

If a point is such that it is the vertex of some P_n at which the interior angle of that P_n is no greater than a right angle, it will be called a point of the set W. Furthermore the set W has no other elements. If the set W exists and has limit points, these limit points are points on K. In this paper we shall prove the following theorems.*

THEOREM 1. *For $r \leqq 1/3$, K has a unique tangent at every point which is not a limit point of W. Hence, except at such limit points, the inclination of the tangent is continuous.*

THEOREM 2. *For $1/3 < r < 1/2$, the right-handed and left-handed tangents to K at an M-point do not coincide.*

THEOREM 3. *For $1/3 < r < 1/2$, if $g_{(i)}$ is the exterior angle between the tangents to K at $M_{(i)}$ which is either an M-point or a limit point of W, then $\Sigma g_{(i)} = 2\pi$. Hence at all other points, K has a unique tangent. The inclination of the tangent is continuous on the set on which it is unique. Also, the linear point set whose elements are the inclinations of the tangents to K (both left-handed and right-handed where different) is a non-dense perfect set of zero measure.*

2. *Tangents at M-points.* Let O be an M-point of P_q. We shall use the symbols O, z_0, $z_0(2)$, $z_0(3)$, $\cdots$, $z_0(t-1)$ to represent

* Some of these results were first obtained by L. Lawrence. See this Bulletin, vol. 39, p. 40, Abstract 39-1-63.

the M-points of P_q in regular counterclockwise order. Similarly, the symbols $O, z_n, z_n(2), z_n(3), \cdots, z_n(2^n t-1)$ will be used to represent the M-points of P_{q+n} in regular counterclockwise order. Thus we may write

$$z_n(s) = z_{n+1}(2s).$$

That vertex of P_{q+n} which is included between $z_n(s-1)$ and $z_n(s)$ will be represented by the symbol $w_n(s)$. We shall write w_n in place of $w_n(1)$. Let O be the origin of coordinates. We now have

$$\begin{aligned} w_{n+1} &= (1-2r)w_n, \\ w_{n+1}(2) &= 2rz_n + (1-2r)w_n, \\ z_{n+1} = \frac{1}{2}(w_{n+1} + w_{n+1}(2)) &= rz_n + (1-2r)w_n. \end{aligned} \tag{1}$$

Now let

$$\begin{aligned} R(n) &= \frac{(1-2r)^{n+1} - (1-2r)r^n}{1-3r}, &&\text{for } r \neq 1/3, \\ R(n) &= (1/3)^n n, &&\text{for } r = 1/3. \end{aligned} \tag{2}$$

We now prove, by induction on n, the relations

$$w_n = (1-2r)^n w_0, \qquad z_n = r^n z_0 + R(n) w_0. \tag{3}$$

Henceforth, we shall let Ow_0 be the x-axis. We shall assume that the interior of P_q is above the x-axis. We note that if K has a unique tangent at an M-point $z_n(s)$ of P_{q+n}, the side $(w_n(s), w_n(s+1))$, of which $z_n(s)$ is the midpoint, must be that tangent. Thus K has a unique tangent at O only if

$$y'_+ = y'_- = 0 \tag{4}$$

at O. But, setting

$$z_n(s) = x_n(s) + iy_n(s), \qquad w_n(s) = u_n(s) + iv_n(s),$$

we have at O,

$$y'_+ = \lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{r^n y_0}{r^n x_0 + R(n)u_0}, \tag{5}$$

$$y'_+ = 0, \qquad \text{at } O,\ r \leqq 1/3. \tag{6}$$

$$y'_+ = \frac{ky_0}{kx_0 + (1-k)u_0}, \qquad \text{at } O,\ r > 1/3,\ k = 3 - 1/r. \tag{7}$$

Results similar to (6) and (7) may be obtained for y_-' at O. Since P_q is an arbitrary polygon of the sequence $\{P_n\}$ and since O is an arbitrary M-point of P_q, Theorem 2 follows as a consequence of the inconsistency of (4) with (7). We may state also the following theorems.

THEOREM 4. *For $r \leqq 1/3$, K has a unique tangent at every M-point. This tangent is the side of P_n of which the M-point is the midpoint. An M-point cannot be a limit point of W in this case.*

THEOREM 5. *For $r > 1/3$, the right-handed tangent to K at O divides each of the half-sides $w_0z_0, \cdots, w_nz_n, \cdots$ in the ratio $k/(1-k)$. A similar result holds for the left-handed tangent at O and for the right-handed and left-handed tangents at every M-point.*

3. *Tangents at Non-M-points*; $r \leqq 1/3$. We shall now complete the proof of Theorem 1. Let z be a non-M-point of K. For every non-negative integer m, there exists an s such that z is on that arc $(z_m(s-1), z_m(s))$ of K which is inscribed in the angle at $w_m(s)$ of P_{q+m}. If z is not a limit point of W, we can find an m so large that the interior angle of P_{q+m} at the corresponding $w_m(s)$ is greater than a right angle. By a proper choice of q and O, we can make $m=0$, $s=1$. Hence the interior angle at w_0 is greater than a right angle. Let

$$c_n(s) = \text{amp}\,(w_n(s+1) - w_n(s)).$$

Now c_0, the exterior angle of P_q at w_0, is less than a right angle. Let a and b, respectively, be the left-handed and right-handed tangents to K at z. In virtue of Theorem 4, Theorem 1 will be proved if it is shown that $a=b$ for $r \leqq 1/3$.

Assume $a \neq b$. Hence $a<b$. Now C, the set of all the values of the $c_n(s)$, has no point interior to the interval (a, b). Let (a', b') be the largest subinterval of $(0, c_0)$ which contains (a, b) and which is such that none of its interior points are points of C. Choose an ϵ such that

$$r(b' - a') > \epsilon > 0.$$

There exists an N so large that the inclinations of a certain pair of successive sides of P_{q+N} differ from a' and b', respectively, by less than ϵ. Without any loss of generality, we may assume that

$N=0$, and that the two sides of P_q whose inclinations differ from a' and b' by less than ϵ are the sides which meet at w_0. Hence

$$\epsilon > a' \geqq 0, \qquad c_0 \geqq b' > c_0 - \epsilon.$$

We now have for every n, either

$$c_0 \geqq c_n \geqq b' \tag{8a}$$

or

$$\epsilon > a' \geqq c_n \geqq 0. \tag{8b}$$

We shall prove that if (8a) holds for $n=1$, (8a) holds for every n. Hence, since

$$c_{n+1} = \text{amp } z_n,$$

we shall have at O

$$y_+' = \lim_{n\to\infty} \tan \text{amp } z_n \geqq \tan b' > 0. \tag{9}$$

This contradicts (6). Likewise, the assumption that (8b) holds for $n=1$ leads to a contradiction of Theorem 4 for the left-handed tangent at z_0.

Assume (8a) for some n. From triangles (O, w_n, z_n) and (O, w_{n-1}, z_{n-1}), we have

$$\begin{aligned} \tan c_{n+1} &= \frac{y_n}{x_n} = \frac{r \sin c_{n-1} \sin c_n}{r \sin c_{n-1} \cos c_n + (1-2r) \sin (c_{n-1} - c_n)} \\ &= \frac{r \tan c_{n-1} \tan c_n}{(1-r) \tan c_{n-1} - (1-2r) \tan c_n} > r \tan c_n > \tan \epsilon. \end{aligned} \tag{10}$$

Comparing this result with (8a) and (8b), we conclude that (8a) holds for every n if (8a) holds for $n=1$. This completes the proof of Theorem 1.

4. *Tangents at Non-M-points*; $r>1/3$. We shall now proceed to prove Theorem 3. Let $a_n(s)$ and $b_n(s)$, respectively, be the inclinations of the left-handed and the right-handed tangents to K at $z_n(s)$. Also let

$$q_n = \sum_{s=1}^{2^n} (a_n(s) - b_n(s-1)).$$

It will suffice to prove that, if $c_0<\pi/2$, then

$$\lim_{n\to\infty} q_n = 0.$$

Noting that the interior angle of P_{q+n} at $w_n(s)$ is less than either of the interior angles of P_{q+n+1} at $w_{n+1}(2s-1)$ and $w_{n+1}(2s)$, we say that it will suffice to prove that

$$q_1 < eq_0, \tag{11}$$

where

$$e = 1 - k^2 \cos^2 c_0, \qquad k = 3 - 1/r.$$

Let $b(0)$ be the inclination of the right-handed tangent to K at O. Recalling Theorem 5, we write

$$\begin{aligned} \tan b(0) &= \frac{k \sin c_1 \sin c_0}{k \sin c_1 \cos c_0 + \sin (c_0 - c_1)}, \\ \tan a_1 &= \frac{h \sin c_1 \sin c_0}{h \sin c_1 \cos c_0 + \sin (c_0 - c_1)}, \end{aligned} \tag{12}$$

where

$$1 > h = r/((1 - 2r)k + r) = 1/((1 - k)k + 1) > k > 0,$$

$$\frac{\tan c_1 - \tan a_1}{\tan c_1 - \tan b(0)} = \frac{1 - h}{1 - k}\cdot\frac{\tan c_0 - (1 - k) \tan c_1}{\tan c_0 - (1 - h) \tan c_1} > k^2. \tag{13}$$

Applying the law of the mean, we get

$$\begin{aligned} &c_1 - a_1 > \cos^2 c_0(\tan c_1 - \tan a_1), \\ &c_1 - b(0) < \tan c_1 - \tan b(0), \\ &a_1 - b(0) = (c_1 - b(0)) - (c_1 - a_1) < e(c_1 - b(0)). \end{aligned} \tag{14}$$

Likewise

$$a_0 - b_1 < e(a_1 - c_1).$$

Adding, we get

$$(a_1 - b(0)) + (a_0 - b_1) < e(a_1 - b(0)), \tag{15}$$

which is precisely (11).

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