AN INVOLUTORIAL LINE TRANSFORMATION*

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1. *Introduction*. Consider a non-singular quadric H, a plane π not tangent to H, and a point O on H but not on π. In the plane π take a Cremona involutorial transformation I_n of order n with fundamental points in general position (not necessarily on the curve of intersection of π and H). Project H from O upon π by the projection P. The point transformation PI_nP^{-1} is involutorial and leaves H invariant as a whole. A point A on $H \sim (P)$ B on π;† $B \sim (I_n)$ B'; $B' \sim (P^{-1})$ A' on H. Now an arbitrary line t, with Plücker coordinates y_i, $(i=1, \cdots, 6)$, meets H in two points A_1, A_2 which $\sim(PI_nP^{-1})$ A_1', A_2'. The line $A_1'A_2' \equiv t'$ shall be called the conjugate of t by the *line transformation* T, and we write $t \sim (T)t'$. Since the point transformation PI_nP^{-1} is involutorial, so will the line transformation T be involutorial.

2. *Order of the Transformation* T. The coordinates of the points A_1, A_2 in which t meets H are quadratic functions of y_i; the coordinates of B_1, B_2 are linear in the coordinates of A_1, A_2 and hence are also quadratic functions of y_i; the coordinates of B_1', B_2' are functions of degree n in the coordinates of B_1, B_2 and are therefore functions of degree $2n$ in y_i; finally A_1', A_2' have coordinates of degree $2n$ in y_i. The Plücker coordinates of a line are quadratic functions of the coordinates of two points which determine the line, and hence the Plücker coordinates x_i of t' are of degree $4n$ in y_i. Thus T is of order $4n$.

3. *The Singular Lines of* T. Denote by O_1, O_2 the points where the generators g_1, g_2 of H through O meet π. The points O_1, $O_2 \sim (I_n) O_1'$, $O_2' \sim (P^{-1}) Q_1$, Q_2. The line $t \equiv Q_1Q_2 \sim (T)$ the entire plane field of lines (g_1g_2), since O_1, $O_2 \sim (P^{-1}) g_1$, g_2.

Any line t tangent to H meets H in two points coincident at A. The point $A \sim (PI_nP^{-1})$ A', and hence $t \sim (T)$ the pencil of tangents to H at A'.

Since $O \sim (P)$ the whole line $O_1O_2 \sim (I_n)$ a curve ρ of order

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† The symbol $\sim(P)$ means "corresponds in the transformation P to."

$n \sim (P^{-1})$ a curve of order $2n$ with an n-fold point at O, any line through O meeting H again at $A \sim (T)$ a cone of order $2n$ with vertex A' and an n-fold generator $A'O$. However, when t is tangent to H at O so that both points of intersection with H coincide there, then $t \sim (T)$ a congruence of lines, bisecants of the curve of order $2n$ into which ρ is projected by P^{-1}. The order of the congruence is the number of bisecants through an arbitrary point of space, and hence the number of apparent double points of the curve. Since ρ is rational and since also its projection on H by P^{-1} is rational, we have, from an arbitrary point of space,

$$\frac{(2n-1)(2n-2)}{2} - \frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{2} = n(n-1)$$

apparent double points, and hence the conjugate congruence is of order $n(n-1)$. The class is the number of bisecants lying in an arbitrary plane, which is $n(2n-1)$.

Denote the regulus to which g_1 belongs by k_1 and that to which g_2 belongs by k_2. A line t belonging to $k_1 \sim (P)$ a line through O_2 which line $\sim (I_n)$ a curve of order $n \sim (P^{-1})$ a curve of order $2n$ on H. Again we find that $t \sim (T)$ a congruence of order $n(n-1)$ and class $n(2n-1)$. So also for any line of the regulus k_2.

The line $t \equiv g_1 \sim (P) O_1 \sim (I_n) O_1' \sim (P^{-1}) Q_1$, and hence $t \sim (T)$ the pencil of tangents to H at Q_1 and likewise $t \equiv g_2(T)$ the pencil of tangents to H at Q_2.

4. *The Invariant Lines of T.* Let the curve of invariant points of I_n be Δ_m of order m and genus p. Then $\Delta_m \sim (P^{-1})$ δ_{2m} of order $2m$ and also of genus p. Any bisecant of δ_{2m} is invariant under T, and hence the invariant lines form a congruence of order $m(m-1)-p$ and of class $m(2m-1)-p$. If I_n has q isolated invariant points $R_1, R_2, \cdots, R_q$, they $\sim (P^{-1})$ q points $S_1, S_2, \cdots, S_q$ on H, and hence there are ${}_qC_2 = q(q-1)/2$ additional invariant lines of T.

5. *Special Cases of T when $n=1$.* Choose I as the harmonic homology with center R and axis Δ. By taking R and Δ in general position in π, we produce the desired results by replacing n by the number one in the foregoing paragraphs. It is only when we choose R and Δ in special positions with regard to O_1, O_2 that the results must be altered.

Let Δ be the line O_1O_2. The order of T is 4. Since each point of Δ is invariant under I, $O_1, O_2 \sim (I)\ O_1, O_2 \sim (P^{-1})\ g_1, g_2$. Hence every line of the plane field $(g_1g_2) \sim (T)$ the whole plane field (g_1g_2).

Any line t through O, meeting H at a second point $A \sim (T)$ the two pencils $A'g_1$, $A'g_2$. A line t tangent to H at $O \sim (T)$ the plane field of lines (g_1g_2).

A line t of the regulus $k_1 \sim (P)$ a line m in π through $O_2 \sim (I)$ another line m' through $O_2 \sim (P^{-1})$ another generator m_1 belonging to k_1, and thus $t \sim (T)$ the plane field (m_1g_2). Likewise a line t belonging to the regulus $k_2 \sim (T)$ an entire plane field of lines.

The entire plane field (g_1g_2) and the bundle (O) are invariant as well as singular under T.

Now choose R at O_1 and Δ in general position in π. Each line through R in π is invariant as a whole under I, and in particular

$$O_1O_2 \sim (I)O_1O_2; \quad O_1 \sim (I)O_1; \quad O_2 \sim (I)B_2'$$

on O_1O_2. Any line t lying in the plane g_1g_2 meets g_1, g_2 in points A_1, A_2 which points $\sim (P)O_1$, $O_2 \sim (I)O_1$, $B_2' \sim (P^{-1})g_1$, O; hence $t \sim (T)g_1$. Since T is involutorial, $g_1 \sim (T)$ the plane field $(g_1g_2)\cdot t \equiv g_2 \sim (T)$ the pencil of tangents to H at O.

Any line t belonging to the regulus $k_2 \sim (T)$ the whole plane field (tg_1). Thus the regulus k_2 is invariant as well as singular under T. Any line t belonging to the regulus $k_1 \sim (P)$ a line m through $O_2 \sim (I)$ a line m' through $B_2' \sim (P^{-1})$ the conic H, Om'. Thus $t \sim (T)$ the plane field (Om').

The invariant lines of T consist of the plane field $(O\Delta)$, the pencil of tangents to H at O, the generator g_1 and the regulus k_2. A like special case arises when we take R at O_2 and Δ in general position in π. The results are readily obtained by interchanging the subscripts 1 and 2 in the discussions in the foregoing paragraphs.

By taking R in general position and Δ through O_1 but not through O_2, we have a third special case of T when $n=1$. Now, the point O_1 is invariant under I but $O_2 \sim (I)B_2'$, and $O_1O_2 \sim (I)O_1B_2' \sim (P^{-1})$ a generator b_2 of the regulus k_2. Thus any line t passing through O and meeting H at $A \sim (T)$ the pencils $A'b_2$, $A'g_1$. Any line t tangent to H at $O \sim (T)$ the plane field (b_2g_1).

Any line t belonging to the regulus $k_2 \sim (P)$ a line m through $O_1 \sim (I)$ another line m' through $O_1 \sim (P^{-1})$ another generator m_2 belonging to the regulus k_2. Thus $t \sim (T)$ the plane field $(m_2 g_2)$. Any line t belonging to the regulus $k_1 \sim (P)$ a line q through $O_2 \sim (I)$ a line q' through $B_2' \sim (P^{-1})$ the conic H, Oq'. Thus $t \sim (T)$ the plane field (Oq').

The invariant lines of T are the plane field $(O\Delta)$ and the line OR. Similarly we have a special case when Δ passes through O_2 and R is in general position.

A fourth special case of T when $n=1$ is found by taking R at O_1 and Δ through O_2. Both O_1 and O_2 are invariant under I but the other points of O_1O_2 are not invariant. A line t through O and meeting H again at $A \sim (T)$ the two pencils $A'g_1$, $A'g_2$. Any line t tangent to H at $O \sim (T)$ the plane field $(g_1 g_2)$.

A line t belonging to $k_2 \sim (T)$ the plane field (tg_1), and a line t belonging to $k_1 \sim (P)$ a line m through $O_2 \sim (I)$ another line m' through $O_2 \sim (P^{-1})$ another generator m_1 of k_1. Thus $t \sim (T)$ the plane field $(m_1 g_2)$.

The invariant lines of T consist of the pencil of tangents to H at O, the plane field $(O\Delta)$, the generator g_1 and the regulus k_2.

By choosing $n > 1$ and taking the F-points, the curve Δ, and the P-curves of I_n in special relation to O_1, O_2, we can set up a limitless number of specializations of this transformation.

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