and between the points $m/2^n$, let f_n be defined linearly.

6. *A Set of Conditions Sufficient to Insure Convergence in Variation.*

THEOREM 7. *Let $f_n(x)$ be a sequence of absolutely continuous functions converging to a limit function $f_0(x)$ on (a, b); let $f_n'(x)$ converge asymptotically to a limit function, and let $f_n'(x)$, $(n=1, 2, 3, \cdots)$, be dominated by a summable function; then we have $f_n(x)$—v→$f_0(x)$ on (a, b).*

It is easily seen that the hypotheses imply (i) that $f_0(x)$ is absolutely continuous, so that we may write

$$T_a^b(f_n) = \int_a^b |f_n'(x)|\, dx, \quad (n = 0, 1, 2, \cdots),$$

and (ii) that we may pass to the limit under the integral sign.

COROLLARY. *Let the series $\sum_{i=0}^{\infty} a_i x^i$, with real coefficients, have the radius of convergence $R(>0)$; let the sum of the series be denoted by $S(x)$, and let $S_n(x) = \sum_{i=0}^{n} a_i x^i$; then we have $S_n(x)$ —v→$S(x)$ on each interval (a, b), $(-R<a<b<R)$.*

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TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH CERTAIN RATIONAL CURVES—COMPOSITE BASIS CURVES*

BY AMOS BLACK

1. *Introduction.* In a preceding paper† the author found and discussed the involutorial transformations belonging to the special complex of lines which meet a rational curve r of order m, $(m=2, 3, 4, 5)$, and having a pencil of invariant cubic surfaces which contain the curve r as a simple basis element, with the restriction that the residual basis curve, γ_{9-m}, of the pencil should not be composite. In this paper we shall discuss the cases where γ_{9-m} is composite.

2. *Equations of the Transformation.* The equations of the

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† *Types of involutorial space transformations associated with certain rational curves*, Transactions of this Society, vol. 34 (1932), pp. 795–810.

transformation and the image of r_m are identical with those of the preceding paper (p. 797), namely:

$$I_{6m+5}: \qquad x_i = y_iR - z_iM, \qquad (i = 1, 2, 3, 4),$$

and

$$M_{3m+5}: \quad r_m^{(m+1)+2t} \; \gamma_{9-m}^{m+1},$$

$$r_m \sim R_{6m+4}: \quad r_m^{(2m+1)+2t} \; \gamma_{9-m}^{2m+1},$$

$$R'_{(m+1)(3m-5)}: \quad r_m^{[(m+1)(m+2)+1]+(m-2)t} \; \gamma_{9-m}^{(m-1)(m-2)},$$

where γ_{9-m} may or may not be composite.

Let γ_{9-m} be composite, and let one part be γ_n, $(1<n<9-m)$. Fix a point P on γ_n. Let L be an arbitrary point on r_m. The line PL meets F_L in P, L, and a third point P', the image of P. As L describes r_m, the line PL generates a cone K_m with one point P' on each generator. Then the locus of P' is a curve of order $m+$ the number of times PL is tangent to F_L at P. Given a point L, the tangent plane of F_L at P intersects r_m in m points K. Conversely, given a point K, the F_3 whose tangent plane at P passes through K is unique; hence there is only one point L. This $(1, m)$ correspondence on r_m has $m+1$ coincidences, and

$$P \sim C_{2m+1}: \quad P^{m+1}.$$

As P traces γ_n, the C_{2m+1} generates a surface Γ which is the image of γ_n.

The surface Γ may be found in an alternate manner. Let O be a fixed point on r_m and P an arbitrary point on γ_n. The line PO cuts F_0 in O, P, and a third point P' which is the part of the image of P which lies on F_0. As P traces γ_n, PO generates a cone K_n, and P' generates a curve C which is the part of the image of γ_n which lies on F_0. The cone K_n and F_0 intersect in γ_n and C. The surface Γ is obtained by eliminating the parameter (λ, μ) between K_n and F_0.

Let us consider a cone K_n. It stands on γ_n and has its vertex at an arbitrary point O (λ, μ) on r_m, and is met by r_m in $n(m-1)-i$ arbitrary points other than O, where i is the number of intersections of r_m and γ_n. Then K_n is met by r_m in one n-fold and $n(m-1)-i$ simple points, hence the parameter (λ, μ) enters the equation of K_n to degree $n+n(m-1)-i=mn-i$. If the parameter is eliminated between $K_n(x, \lambda, \mu)$: γ_n, and

$$F_0 = \mu F_3 - \lambda F_3': \quad r_m \gamma_n \gamma_{9-m-n},$$

we have

$$\Gamma(x, F_3, F_3')$$

of degree n in (x), and of degree $mn-i$ in F_3, F_3'. Hence

$$\Gamma_{3(mn-i)n}: \quad r_m^{mn-i} \gamma_n^{mn-i+1} \gamma_{9-m-n}^{mn-i}.$$

Since $C_{2m+1}: P^{m+1}$ is perspective from P, then P is invariant in the $m+1$ directions of the tangents of C_{2m+1} at P. Thus $m+1$ sheets of Γ touch the $m+1$ sheets of M along γ_n.

From the table of images and intersections we find that n sheets of Γ have a common tangent plane at all points O of r_m, the tangent plane of F_0 at O.

There is one set of transformations which involves additional explanation; namely, that where $m=2$ and the residual γ_9 is composite, consisting of a straight line which meets r_2 twice and a sextic where the sextic may, or may not, be composite. We shall treat the case where the sextic is not composite.

3. *Equations and Images.* The fundamental curves are r_2, γ_1, γ_6, $(p=4)$, where $[r_2, \gamma_1]=2$ points, $[r_2, \gamma_6]=4$ points, and $[\gamma_1, \gamma_6]=2$ points. We see that γ_1, r_2 forms a complete plane section of any F_3. If we choose P as a general point of this plane, the particular F_3 determined by P is composite and has the plane for one component. Thus the whole line PO lies on F_3, hence is parasitic. But there is a pencil of such lines in the plane and the plane factors out of the transformation. The degree of the transformation is reduced by one, hence is sixteen. The plane is also a factor of R and M.

$$\text{A plane} \sim S_{16}: \quad r_2^{4+3t} \gamma_1^4 \gamma_6^5,$$

$$r_2 \sim R_{15}: \quad r_2^{4+2t} \gamma_1^4 \gamma_6^5,$$

$$M_{10}: \quad r_2^{2+2t} \gamma_1^2 \gamma_6^3.$$

Fix a point O on r_2. The line OL, where L is an arbitrary point on r_2, meets F_L in a point P' on γ_1. As L describes r_2, P' generates γ_1 which is the image of O. But as O describes r_2, γ_1 remains fixed, hence there is no surface R'.

The line joining a fixed point P on γ_1 to an arbitrary point O

on r_2 meets F_0 in a point P' on r_2. As O describes r_2, P' also describes r_2. However, when O is either of the points of intersection of γ_1, r_2, the point P' can not only be a point of r_2 but also any point on γ_1. Thus the image of any point P on γ_1 is r_2, and γ_1 counted twice. But as P traces γ_1, the image of P remains fixed; hence there is no surface which is the image of γ_1.

The two curves r_2, γ_1 each play a dual role. The curve r_2 is a fundamental curve of the first species with image R_{15}, and a fundamental curve of the second species with image γ_1. The line γ_1 is a fundamental line of the second species with image r_2 and a parasitic line.

The reduction in the degree of the transformation does not affect the image of γ_6; hence

$$\gamma_6 \sim \Gamma_{30}: \quad \overset{8+6t}{r_2} \; \overset{8}{\gamma_1} \overset{9}{\gamma_6}.$$

The Jacobian is $J_{60} = R_{15}^2 \Gamma_{30}$.

4. *Number of Parasitic Lines.* We have already seen that γ_1 is itself a parasitic line counted twice. We shall think of it as two parasitic lines. In a manner similar to that of the preceding paper (pp. 798–800) we find twenty-two other parasitic lines each of which meets r_2 once and γ_6 twice. In all there are twenty-four parasitic lines, which is five less than the number in the case for γ_7 non-composite.

The procedure in this and the preceding paper can be immediately generalized to spaces of higher dimensions. One interesting feature of the generalization is that images of lines, planes, three-way spaces, etc., must all be considered. No essentially new ideas are introduced.

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