ON SIMPLY TRANSITIVE PRIMITIVE GROUPS*

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Certain properties of the transitive constituents of the subgroup that fixes one letter of a simply transitive primitive permutation group will be analyzed in this paper. We shall denote the simply transitive primitive group by G and its subgroup that fixes the letter x, say, by $G(x)$.

THEOREM 1. *If $G(x)$ has two transitive constituents of relatively prime degrees m and n, $n > m$, it has a transitive constituent of degree $(>n)$ a divisor of mn.*

Let $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_n$, respectively, be the letters of two transitive constituents of $G(x)$ of relatively prime degrees m and n, and assume $n > m$. Let us denote the order of the group $G(x)$ by g. Now the subgroup $G(x)(a_1)$ of $G(x)$, which fixes the letter a_1, is of order g/m, while the subgroup $G(x)(a_1)(b_1)$ of $G(x)(a_1)$, which fixes the letter b_1, is of order $g/(mt)$, $(1 \leqq t \leqq n)$. Using the same notation, we have on the other hand g/n as the order of the group $G(x)(b_1)$, and $g(ns)$, $(1 \leqq s \leqq m)$, as the order of $G(x)(b_1)(a_1)$. Hence $ns = mt$. Since m and n are relatively prime and $s \leqq m$ and $t \leqq n$, we have $s = m$ and $t = n$. Thus in the group $G(a_1)(x)$, which is identical with $G(x)(a_1)$, the n letters $b_1, b_2, \cdots, b_n$ are permuted transitively. Now since the subgroup that fixes one letter of a simply transitive primitive group is a maximal subgroup, $G(x)$ and $G(a_1)$ cannot both have a transitive constituent on the same letters. Consequently, in $G(a_1)$ the letters $b_1, b_2, \cdots, b_n$ belong to a transitive constituent of degree $q > n$. In $G(a_1)$ the letter x belongs to a transitive constituent of degree m. Hence $G(a_1)(b_1)(x)$ is of order $g/(qv)$, where $1 \leqq v \leqq m$. But the order of the group $G(x)(a_1)(b_1)$ is $g/(mn)$. Hence $qv = mn$ and q divides mn.

The following corollaries are immediate consequences of this theorem.

COROLLARY 1. *If $G(x)$ has exactly two transitive constituents, the degrees of the two transitive constituents have a common factor greater than one.*

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COROLLARY 2. *The degree of a transitive constituent of maximum degree of $G(x)$ has a factor in common with the degree of each of the transitive constituents of $G(x)$.*

We turn now to an analysis of $G(x)$ when it contains a regular constituent.

THEOREM 2. *Let $G(x)$ have a regular constituent M of degree m and let the order of $G(x)$ exceed m. Denote by H the invariant subgroup corresponding to the identity of M. Then*

(1) *the transitive constituent M', with which M is paired in $G(x)$, has an invariant subgroup, consisting of all its permutations that are in H, with transitive constituents of degree t, $(2 \leqq t \leqq m)$;*

(2) *the degree of no transitive constituent of H contains a factor prime to t;*

(3) *H has transitive constituents of degree less than t;*

(4) *the degrees of the transitive constituents of H of degree less than t have no common factor.*

We shall use the notation of the previous theorem. Let the regular constituent M of degree m be on the letters $a_1, a_2, \cdots, a_m$. Let the letters which H displaces be Greek letters and those which it fixes be italic letters. Note that since M is regular, the subgroup $G(x)(a_1)$ is H.

The transitive constituents of the subgroup that fixes one letter of a transitive group occur in pairs of equal degrees.* Two members of a pair may coincide and then the transitive constituent is said to be paired with itself. If M is paired with itself, the permutation $S=(xa_1)\cdots$ exists in G and transforms $G(x)(a_1)$, which is H, into itself; but since $G(x)$ is a maximal subgroup, it is the largest subgroup of G in which H is invariant. Thus M is not paired with itself.† Then let M be paired with a transitive constituent on italic letters. The permutation $S=(b_1xa_1\cdots)\cdots$, which exists because of this pairing, transforms H into a subgroup, fixing both x and a_1, which consequently is H itself. We conclude that M must be paired with a transitive constituent on Greek letters. Hence there exists the permutation $S=(\alpha x a_1\cdots)\cdots$, where α is a letter of the

* Burnside, Proceedings of the London Mathematical Society, vol. 33 (1901), p. 162.

† Manning, Transactions of this Society, vol. 29 (1927), p. 815, §§1 and 9.

transitive constituent M' which is paired with M. Now $SG(x)(a_1)\, S^{-1}=G(\alpha)(x)$ and hence $S^{-1}G(\alpha)(x)S=H$.

We now make an analysis of the degrees of the transitive constituents of H. Let the letters of M' be found in transitive constituents of degree t. We first show that H cannot have transitive constituents whose degree contains a factor prime to t. Of course, if $t=m$ this statement is a repetition of Jordan's theorem that if a prime divides the order of $G(x)$, it divides the order of every constituent group of $G(x)$. Let β be a letter of a transitive constituent of H of degree v, containing a factor v' prime to t. Consider $H(\alpha)$, the subgroup of H which fixes α. Its order is h/t, where h is the order of H. The order of $H(\alpha)(\beta)$ is $h/(tk)$, where k is the degree of the transitive constituent to which β belongs in $H(\alpha)$. On the other hand, the order of $H(\beta)(\alpha)$ is $h/(vr)$, r being the degree of the transitive constituent to which α belongs in $H(\beta)$. Hence $tk=vr$, $k=vr/t$, and k is a multiple of v'. Consequently, every letter of a transitive constituent of H whose degree is divisible by v' is displaced in $H(\alpha)$ in a transitive constituent of degree a multiple of v'. Since $H(\alpha)$ is in H and in $G(x)(\alpha)$, and since $G(x)(\alpha)$ transforms H into itself, $H(\alpha)$ is invariant in $G(x)(\alpha)$. Hence all the letters of transitive constituents of H of degree involving a factor v' are displaced in $G(x)(\alpha)$ in transitive constituents whose degree is divisible by v'. Since H and $G(x)(\alpha)$ are conjugates, each has the same number of transitive constituents whose degrees are divisible by v'. We have just shown that H and $G(x)(\alpha)$ have these transitive constituents on the same letters. Recall that $SG(x)S^{-1}=G(\alpha)$ and $SHS^{-1}=G(\alpha)(x)$. Hence $G(\alpha)$ in which $G(\alpha)(x)$ is invariant displaces these letters in transitive constituents containing no other letters. Thus the group $\{G(\alpha),\ G(x)\}$ permutes these letters among themselves, but $G(x)$ is a maximal subgroup, and, consequently, H has no transitive constituents whose degrees contain a factor prime to t.

Further we show that transitive constituents of degrees less than t exist, and that their degrees have no common factor. The subgroup that fixes one letter of the constituent M' displaces all the other letters of the constituent.* Hence $G(x)(\alpha)$ has transitive constituents on the letters of the constituent M'

* Rietz, American Journal of Mathematics, vol. 26 (1904), p. 9.

of degree less than t. If the transitive constituents of degrees less than t are of degrees $w_1, w_2, \cdots, w_s$ then we shall have $t=k_1w_1+k_2w_2+\cdots+k_sw_s+1$, where at least one $k_i>0$, for S replaces α by x and the letters $\alpha_1, \alpha_2, \cdots, \alpha_{t-1}$ of the transitive constituent of degree t to which α belongs in H by letters of transitive constituents of H of degrees less than t. This is evident when we recall that $S^{-1}G(x)(\alpha)S=H$. Hence a common factor of $w_1, w_2, \cdots, w_s$ is prime to t, contrary to the analysis in the preceding paragraph.

COROLLARY. *The number t defined in Theorem 2 is not a power of a prime.*

Since the degree of no transitive constituent of H is prime to t, and since the transitive constituents of degrees less than t exist and have no common factor, we conclude that t is not a power of a prime.

We use Theorem 2 to prove the following theorems.

THEOREM 3. *If $G(x)$ has a regular constituent of degree pq, p and q primes, it is of order pq.*

Assume the order of $G(x)$ to exceed pq. We follow the notation of Theorem 2. Since the constituent M' is of order p^xq^y, it is solvable,* and since its degree is not a power of a prime, it is not primitive. Let p be less than q. Since the subgroup $G(x)(\alpha)$, that fixes one letter of M', displaces the remaining $m-1$ letters of M', the systems of imprimitivity of M' cannot be of degree p, for then this subgroup has transitive constituents of degrees $\leqq p-1$, but by (2) and (3) of Theorem 2, p is the smallest degree of a transitive constituent of H. Thus M' has systems of imprimitivity of q letters permuted according to a group of degree p, and, since $p<q$, of order p. Corresponding to the identity of the group of the systems, M' has an invariant subgroup of order $p^{x-1}q^y$ which fixes the systems of imprimitivity. The subgroup that fixes one letter of M' fixes one system and hence every system of imprimitivity of M'. Hence it is contained in the invariant subgroup of order $p^{x-1}q^y$, and since its order is $p^{x-1}q^{y-1}$, it is one of at most q conjugates under M'. However,

* Burnside, Proceedings of the London Mathematical Society, (2), vol. 2 (1904), p. 388.

the subgroup that fixes one letter of M' is necessarily one of pq conjugates under M', for it fixes only one letter of M'. Hence $G(x)$ is of order pq.

THEOREM 4. *If $G(x)$ has no more than four transitive constituents, one of which is a regular constituent of degree m, it is of order m.*

We again use Theorem 2 and assume the order of $G(x)$ to exceed m. If the transitive constituents of H of degree less than t are all of the same degree v, say, v is prime to t, for then $t=kv+1$. Hence the transitive constituents of H of degree less than t are of at least two different degrees. Thus H must displace letters of at least three transitive constituents of $G(x)$ for all the transitive constituents of H arising from one transitive constituent of $G(x)$ are of the same degree. If $G(x)$ has no more than four transitive constituents, H has transitive constituents of degree t arising only from the transitive constituent M' of $G(x)$. Now $G(x)(\alpha)$ displaces the letters of M in transitive constitutents of degree t, for $G(x)(a_1)(\alpha)$ is of order $g/(mt)$, while $G(x)(\alpha)(a_1)$ is of order $g/(mk)$. Thus $k=t$, and since M is regular, every transitive constituent on the letters of M is of degree t in $G(x)(\alpha)$. Since $S^{-1}G(x)(\alpha)S=H$, S replaces every a by an α, where $\alpha, \alpha_1, \cdots, \alpha_{m-1}$ are the letters of M'. Consequently, $S^{-1}G(x)S=G(a_1)$ has a transitive constituent on the letters $\alpha, \alpha_1, \cdots, \alpha_{m-1}$, but the group $\{G(x), G(a_1)\}$ is G. Hence if H is to exist, $G(x)$ must contain more than four transitive constituents.

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