A NOTE ON NILPOTENT ALGEBRAS
IN FOUR UNITS*

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1. *Introduction.* In volume 9 of the Transactions of this Society R. B. Allen gave, without proof, a classification of all associative nilpotent algebras in $n \leq 4$ units.† These results were later verified by A. A. Albert for $n \leq 3$.‡

I have recently completed a reclassification of nilpotent algebras in four units and have discovered several serious errors in Allen's results. Although he stated that his classification separated the algebras into classes of non-equivalent and non-reciprocal algebras, he actually did not accomplish this, as he gave several superfluous cases. Moreover, he erroneously listed certain classes of non-associative algebras which must be replaced by similar classes of associative algebras.

In §2 I shall prove the validity of the following revision of Allen's 16 classes (labelled A) into my §9 classes (labelled G). Allen's classes of *irreducible algebras* are

(1A) $e_1^2 = ae_4, \quad e_2e_3 = be_4 = e_3^2, \quad e_3e_1 = e_4 = -e_3e_2,$

(2A) $ae_1^2 = e_2e_1 = e_2^2 = e_3e_2 = be_3^2 = e_4,$

(3A) $e_1^2 = e_3e_2 = e_4,$

(4A) $ae_1^2 = e_2e_1 = e_2^2 = be_3^2 = e_4,$

(5A) $ae_1^2 = e_2^2 = be_3^2 = e_4,$

(6A) $e_1^2 = e_3, \quad e_2e_1 = e_4, \quad e_1e_2 = e_3 + ae_4, \quad e_2^2 = be_3 + ce_4,$

(7A) $e_1^2 = e_3, \quad e_2e_1 = e_4, \quad e_2^2 = ae_3 + e_4,$

(8A) $e_1^2 = e_3, \quad ae_2^2 = e_2e_1 = e_4,$

(9A) $e_1^2 = e_3, \quad e_2^2 = e_4,$

(10A) $e_1^2 = e_3, \quad e_1e_2 = ae_2e_1 = e_2^2 = e_4,$

(11A) $e_1^2 = e_3, \quad e_1e_2 = ae_2e_1 = e_4,$

(12A) $e_1^2 = ae_2^2 = e_3, \quad e_2e_1 = ae_1e_2 = e_4,$

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* Presented to the Society, October 28, 1933.

† Transactions of this Society, vol. 9, pp. 213.

‡ In his master thesis, pp. 5–7.

§ As may be observed, my own classes are but minor revisions of Allen's classes. These revisions are necessary in order that all algebras may be included.

(13A) $e_1^2 = e_3,$ $e_2^2 = ae_3 + e_4,$ $e_1e_2 = ae_4,$ $e_2e_1 = e_4,$

(14A) $e_1^2 = e_3,$ $e_1e_3 = ae_2e_1 = e_2^2 = e_3^2 = e_4,$

(15A) $e_1^2 = e_3,$ $e_1e_3 = e_4 = e_2e_1,$

(16A) $e_i = e_1^i,$ $(i = 1, 2, 3, 4),$

with constants a, b, c in the reference field F, while my own classes are

  (1G) Same as (1A),

  (4G) $e_2e_1 = e_2^2 = e_4,$ $e_1^2 = ae_4,$ $e_3^2 = be_4,$

  (5G) $e_2^2 = e_4,$ $e_1^2 = ae_4,$ $e_3^2 = be_4,$

  (6G) Same as (6A),

  (9G) Same as (9A),

 (12G) $e_1^2 = e_3,$ $e_2^2 = ae_3,$ $e_1e_2 = e_2e_1 = e_4,$

 (14G) $e_1^2 = e_3,$ $e_1e_3 = e_3e_1 = e_2e_1 = e_4,$ $e_2^2 = ae_4,$

 (15G) $e_1^2 = e_3,$ $e_1e_3 = e_3e_1 = e_4,$ $e_2^2 = ae_4,$

 (16G) Same as (16A).

2. *The Revision.* The classes (14A) and (15A) obviously do not belong in a table of associative algebras. For, if $e_1^2 = e_3$, then $e_1^3 = e_1e_3 = e_3e_1$. Hence, if $e_1e_3 \neq 0$, then $e_3e_1 \neq 0$ and further $e_3^2 = e_1^4 = e_1e_1^3 = e_1(e_1e_3) = e_1a_{13}e_4 = 0$. Both of these conclusions contradict the tables numbered (14A) and (15A).

Class (3A) is superfluous and may be shown to be a sub-case of class (4G). For in class (3A) we have

(1) $$e_1^2 = e_3e_2 = e_4,$$

or, on replacing e_3 by e_2, e_2 by e_1, and e_1 by e_3,

(2) $$e_3^2 = e_2e_1 = e_4.$$

Put $e_2' = e_1 + e_2$ and obtain

(3) $$e_2'^2 = e_2'e_1 = e_3^2 = e_4,$$

which is class (4G) for $a = 0$ and $b = 1$.

Class (7A) is a special case of class (6A). For in class (7A)

(4) $e_1^2 = e_3,$ $e_2e_1 = e_4,$ $e_2^2 = e_4 + de_3.$

Put $e_2' = e_1 + e_2$ and obtain

(5) $\begin{cases} e_1^2 = e_3, \quad e_1e_2' = e_1^2 + e_1e_2 = e_3, \quad e_2'e_1 = e_1^2 + e_2e_1 = e_3 + e_4, \\ e_2'^2 = e_1^2 + e_1e_2 + e_2e_1 + e_2^2 = 2e_4 + (d+1)e_3. \end{cases}$

Let $(e_3+e_4)=e_4'$, so that

(6) $\quad e_1^2 = e_3, \quad e_2'e_1 = e_4', \quad e_1e_2' = e_3, \quad e_2'^2 = (d-1)e_3 + 2e_4'$,

which is class (6A) for $a=0$, $b=d-1$, and $c=2$.

We may also omit class (8A). Since e_4 is a basal unit, $a\neq 0$. Then if $b=(1/a)$, class (8A) becomes

(7) $\qquad\qquad e_1^2 = e_3, \qquad e_2e_1 = e_4, \qquad e_2^2 = be_4.$

Let $be_4=e_4'$ and put $e_1=(e_1'/b)$ and let $b^2e_3=e_3'$, so that

(8) $\qquad\qquad e_2^2 = e_2e_1' = e_4', \qquad e_1'^2 = e_3'.$

Replace e_2 by $e_2'=(e_1'+e_2)$ to give

(9) $\quad e_1'^2 = e_3', \quad e_1'e_2' = e_3', \quad e_2'e_1' = e_3' + e_4', \quad e_2'^2 = e_3' + 2e_4'.$

Let $(e_3'+e_4')=e_4''$ and obtain

(10) $\quad e_1'^2 = e_3', \quad e_1'e_2' = e_3', \quad e_2'e_1' = e_4'', \quad e_2'^2 = -e_3' + 2e_4'$,

which is class (6A) for $a=0$, $b=-1$ and $c=2$.

Class (10A) likewise involves no new algebras. Algebras (10A) have the form

(11) $\qquad\qquad e_1^2 = e_3, \qquad e_1e_2 = ae_2e_1 = e_2^2 = e_4,$

and again $a\neq 0$, since e_4 is a basal unit. If $a=1$, replace e_1 by $e_1'=e_1-e_2$ and let $(e_3-e_4)=e_3'$, so that

(12) $\qquad\qquad\qquad e_1'^2 = e_3', \qquad e_2^2 = e_4,$

while all other products vanish. This is class (9A).

Henceforth, in class (10A) assume $a\neq 0, 1$ and let $b=1/a$, so that algebras (10A) become

(13) $\quad e_1^2 = e_3, \qquad e_1e_2 = e_2^2 = e_4, \qquad e_2e_1 = be_4, \qquad (b\neq 0, 1).$

Let $(e_1+e_2)\,r=e_2'$, where $r=b/(b-1)$; then

(14) $\quad \begin{cases} e_1e_2' = r(e_3 + e_4), \qquad e_2'e_1 = r(e_3 + be_4), \\ e_2'^2 = r^2(e_3 + e_4 + be_4 + e_4), \qquad e_1^2 = e_3. \end{cases}$

Let $r(e_3+be_4)=e_4'$; then $e_4=(e_4'-re_3)/(br)$. Hence (14) becomes

$$e_1^2 = e_3, \qquad e_2' e_1 = e_4',$$

$$(15) \quad e_1 e_2' = re_3 + \frac{r(e_4' - re_3)}{rb} = re_3\left(1 - \frac{1}{b}\right) + \frac{e_4'}{b} = e_3 + \frac{e_4'}{b},$$

$$e_2'^2 = Ae_3 + Be_4, \qquad\qquad\qquad (A \text{ and } B \text{ in } F).$$

This is evidently class (6A). Thus we have shown that algebras (10A) are transformable either into algebras (6A), or into algebras (9A).

The class of algebras (11A) likewise contains only algebras that are included in other classes. Algebras (11A) are of the form

$$(16) \qquad\qquad e_1^2 = e_3, \qquad e_1 e_2 = a e_2 e_1 = e_4,$$

and $a \neq 0$, since e_4 is a basal unit. If $a = 1$, (16) represents commutative algebras which are included in my class (12G) for $a = 0$.

Henceforth in class (11A), assume $a \neq 0, 1$. Algebras (11A) become

$$(17) \qquad e_1^2 = e_3, \qquad e_1 e_2 = e_4, \qquad e_2 e_1 = b e_4, \qquad (b = 1/a).$$

Put $(e_1 + e_2)r = e_2'$, where $r = b/(b-1)$. Then

$$(18) \qquad e_2' e_1 = r(e_3 + b e_4), \qquad e_1 e_2' = r(e_3 + e_4).$$

Let $r(e_3 + b e_4) = e_4'$. Then $e_4 = (e_4' - re_3)/(br)$ and the algebras are of the form

$$e_1^2 = e_3, \qquad e_2 e_1 = e_4', \qquad e_2^2 = ae_3 + Be_4,$$

$$(19) \quad e_1 e_2 = re_3 + \frac{r(e_4' - re_3)}{br} = \frac{r(b-1)e_3}{b} + \frac{e_4'}{b} = e_3 + \frac{e_4'}{b},$$

which is class (6A). Hence algebras (11A) include only algebras contained in other classes.

Let us next examine algebras (12A) which are of the form

$$(20) \qquad\qquad e_1^2 = a e_2^2 = e_3, \qquad e_2 e_1 = a e_1 e_2 = e_4,$$

and $a \neq 0$, since e_3 and e_4 are basal units. If $a = 1$, we have class (12G) for $a = 1$ of my revised table. Henceforth assume $a \neq 0, 1$. Class (12A) then has the form

$$(21) \quad e_1^2 = e_3, \ e_2^2 = Ae_3, \ e_2 e_1 = e_4, \ e_1 e_2 = Ae_4, \quad (A = 1/a \neq 1, 0).$$

Put $e_2' = r(e_1 + e_2)$, where $r = 1/(1-A)$, so that

(22) $e_1 e_2' = r(e_3 + A e_4)$,   and $e_2' e_1 = r(e_3 + e_4)$.

Let $(e_3 + e_4)r = e_4'$, and then $e_4 = (e_4' - re_3)/r$ and

$$e_1^2 = e_3, \qquad e_2 e_1 = e_4', \qquad e_2^2 = Be_3 + De_4,$$

(23)
$$e_1 e_2 = re_3 + \frac{rA(e_4' - re_3)}{r} = e_3 + Ae_4',$$

which is class (6A). Hence class (12A) as given by Allen is superfluous except when $a = 1$.

Class (13A) can be shown unnecessary. It has the form

(24) $e_1^2 = e_3,$ $e_2^2 = ae_3 + e_4,$ $e_1 e_2 = ae_4,$ $e_2 e_1 = e_4.$

If $a = 0$, the algebras (24) are easily transformable to class (6A) (see algebras of class (8A) above). If $a = 1$,

(25) $e_1^2 = e_3,$ $e_2^2 = e_3 + e_4,$ $e_1 e_2 = e_2 e_1 = e_4.$

Put $e_2' = e_1 - 2e_2$ and let $e_3 - 2e_4 = e_4'$, so that

(26)
$$e_1^2 = e_3, \quad e_2'^2 = e_1^2 - 2e_1 e_2 - 2e_2 e_1 + 4e_2^2 = e_3 - 2e_4 - 2e_4$$
$$+ 4e_4 + 4e_3 = 5e_3, \quad e_1 e_2' = e_2' e_1 = e_3 - 2e_4 = e_4',$$

which is class (12G).

Henceforth in class (13A), assume $a \neq 0, 1$. Put $e_1' = e_1 + e_2$ and $e_2' = re_2$, where $r = 1/(a^2 - a)$. Then

$$e_1'^2 = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2 = e_3(1 + a) + e_4(2 + a),$$

(27) $e_1' e_2' = rae_3 + r(a + 1)e_4,$ $e_2' e_1' = 2re_4 + are_3,$

$$e_2'^2 = r^2 ae_3 + r^2 e_4.$$

Pass to the reciprocal algebra and let

$(e_3(1 + a) + e_4(2 + a)) = e_3'$ and $(rae_3 + r(a + 1)e_4) = e_4'$.

Then

$$e_3 = \frac{\begin{vmatrix} e_3' & 2 + a \\ e_4' & r(a + 1) \end{vmatrix}}{r \begin{vmatrix} 1 + a & 2 + a \\ a & 1 + a \end{vmatrix}} = \frac{r(a + 1)e_3' - (2 + a)e_4'}{r},$$

and similarly

$$e_4 = \frac{e_4'(1 + a) - e_3' ra}{r}.$$

Hence our algebras have the form

$$e_1^2 = e_3', \qquad e_2 e_1 = e_4',$$

$$(28) \quad e_1 e_2 = \frac{2r(e_4'(1+a) - e_3' ra)}{r} + \frac{ar(r(a+1)e_3' - (2+a)e_4')}{r}$$

$$= re_3'(a^2 - a) + e_4'(2 - a^2) = e_3' + e_4'(2 - a^2),$$

$$e_2^2 = Be_3' + De_4', \qquad\qquad (B \text{ and } D \text{ in } F).$$

These are again algebras (6A). We have now shown that class (13A) contains only algebras contained in class (6A) or in class (12G).

Class (2A) is also superfluous, as it can be transformed either into class (1A) or into class (4G) as follows. Since e_4 is a basal unit, $a \neq 0$, and $b \neq 0$. Hence class (2A) has the form

$$(29) \quad e_1^2 = a_1 e_3, \qquad e_2^2 = e_2 e_1 = e_3 e_2 = e_4, \qquad e_3^2 = b_1 e_4.$$

If $a_1 = b_1 = 0$, replace e_3 by $e_3' = e_1 + e_3$ to give

$$(30) \qquad\qquad e_2^2 = e_2 e_1 = e_3 e_2 = e_2 e_3 = e_4,$$

or, on interchanging e_3 and e_2,

$$(31) \qquad\qquad e_3^2 = e_2 e_3 = e_4, \qquad e_3 e_1 = e_4 = e_3 e_2.$$

Put $e_1' = -e_1$ and $-e_4 = e_4'$ and obtain class (1A) for $a = 0$ and $b = -1$.

Now assume $a_1 \neq 0$, $b_1 \neq 0$, $a_1 \neq -b_1$. Then the non-singular transformation

$$(32) \quad e_1' = -\frac{1}{a_1} e_1 + e_2, \qquad e_2' = -\frac{b_1}{a_1} e_1 + e_3, \qquad e_3' = e_1 + e_3$$

yields algebras of the form (33) when applied to class (2A):

$$(33) \quad e_1'^2 = Ae_4', \qquad e_2'^2 = e_4', \qquad e_3'^2 = Be_4', \qquad e_2' e_1' = Me_4',$$

where $e_4' = (b_1^2/a_1 - b_1)e_4$ and A, B and M are easily calculated numbers of the field F. The form (33) is readily transformable to class (4G).

If $a_1 \neq 0$, $b_1 = 0$, the above transformation (32) gives

$$(34) \qquad\qquad e_1'^2 = e_4, \qquad e_3'^2 = a_1 e_4, \qquad e_2' e_1' = e_4,$$

or, interchanging e_1' and e_2',

(35) $$e_2'^2 = e_4, \qquad e_3'^2 = a_1 e_4, \qquad e_1' e_2' = e_4.$$

Passing to the reciprocal algebra, we have class (4G) for $a_1 = 0$.

If $a_1 = 0$ and $b_1 \neq 0$, interchanging e_1 and e_3 and passing to the reciprocal algebra, we again have the case $a_1 \neq 0$, $b_1 = 0$. If $a_1 = -b_1$ the transformation (32) is singular. Transformation (36), however, can be used to transform class (2A) with $a_1 = -b_1$ into class (2A) for $a_1 \neq -b_1$, unless $a_1^2 = 2$:

$$e_1' = -2e_1 + e_2 - 2e_3,$$

(36) $$e_2' = \left(\frac{-3a_1 - 1}{2}\right) e_1 + \left(\frac{3a_1 - 1}{2}\right) e_2 + \left(\frac{2 - 3a_1}{2}\right) e_3,$$

$$e_3' = e_1 + e_2 + e_3.$$

This non-singular transformation gives, when applied to class (2A) for $a_1 = -b_1$,

(37)
$$e_1'^2 = -3e_4, \qquad e_3'^2 = 3e_4, \qquad e_2'^2 = \frac{9a_1^2}{4} e_4,$$

$$e_2' e_1' = \frac{3}{2} e_4, \qquad e_3' e_2' = \frac{3a_1^2 - 3}{2} e_4.$$

Let us set $e_1' = (3/2)(4/(9a_1^2)) e_1''$, so that $e_2' e_1'' = (9a_1^2/4) e_4$ and $e_1''^2 = -(27/4) a_1^4 e_4$. Let $e_3' = [(3a_1^2 - 3)/2][4/(9a_1^2)] e_3''$, so that $e_3'' e_2' = (9a_1^2/4) e_4$ and $e_3''^2 = [(2(a_1^2 - 1)/(3a_1^2))^2 (3)] e_4$. If we set $(9a_1^2/4) e_4 = e_4'$, we obtain

(38) $$e_2'^2 = e_3'' e_2' = e_2' e_1'' = e_4', \qquad e_1''^2 = A e_4', \qquad e_3''^2 = B e_4',$$

where $A \neq -B$. Hence we can now use (32) to carry algebras (2A) into algebras (4A).

Assume now for class (2A), $a_1 = -b_1$ and $a_1^2 = b_1^2 = 2$ and apply

$$e_1' = (-a_1 - 1)e_1 + e_2 + (-a_1 - 1)e_3,$$

(39) $$e_2' = 2a_1(a_1 + 1)e_1 - (2a_1^2 + a_1 - 1)e_2 + (2a_1^2 - 1)e_3,$$

$$e_3' = a_1 e_1 + e_2 + a_1 e_3.$$

This non-singular transformation gives

$$e_1'^2 = (-2a_1 - 1)e_4,$$

(40)
$$e_2'^2 = (4a_1^5 + 8a_1^4 - 3a_1^2 + a_1 + 1)e_4 = Ae_4,$$

$$e_3'^2 = (2a_1 + 1)e_4, \qquad e_2' e_1' = (-2a_1)e_4,$$

$$e_3' e_2' = (2a_1 + 1)e_4, \qquad e_1' e_3' = e_3' e_1' = e_2' e_3' = e_1' e_2' = 0.$$

In (40), $a_1 = 2^{1/2}$ or $a_1 = -2^{1/2}$ does not cause any of the parameters to vanish nor does it make the transformation (39) singular. Let $e_1' = (-2a_1/A)e_1''$, $e_3' = [(2a_1+1)/A]e_3''$, $Ae_4 = e_4'$, so that

(41)
$$e_2' e_1'' = e_3'' e_2' = e_2'^2 = e_4', \quad e_1''^2 = \frac{A^2(-2a_1-1)}{(2a_1)^2} \frac{e_4'}{A} = Me_4',$$

$$e_3''^2 = \frac{A^2(2a_1+1)}{(2a_1+1)^2 A} e_4' = Ne_4',$$

where it is quite obvious that $M \neq -N$. We can now use (32) to carry algebras (2A) into algebras (4G).

We have thus shown that classes (2A), (3A), (7A), (8A), (10A), (11A), and (13A) are superfluous and can be omitted. Classes (14A) and (15A) are erroneous and have been replaced by classes (14G) and (15G) which arise naturally in the process of classification. Classes (4A), (5A), and (12A) have been altered so as to allow the parameters to take the value zero which is necessary if all algebras are to be included. The classes of algegras, in my table, with the exception of (1G), (4G), (9G), and (12G), are non-equivalent and non-reciprocal regardless of the field F or special values of the constants in the algebras. Algebras of (12G) are equivalent to algebras of (9G) if and only if a is the square of a number of F. Algebras in classes (1G) and (4G) also may be equivalent if the field and values of a and b are such that certain equations given in my master dissertation are satisfied.

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