THE PROBABILITY LAW FOR THE SUM OF n INDEPENDENT VARIABLES, EACH SUBJECT TO THE LAW $(1/(2h))\operatorname{sech}(\pi x/(2h))$*

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1. *Introduction.* Let the probability of selecting the chance real variable x from the interval $(x, x+dx)$ be to within infinitesimals of higher order, the quantity $(1/(2h)) \operatorname{sech} (\pi x/(2h))dx$. This hyperbolic secant probability or frequency function has been used by others. Roa considered this function in many details as a generating function for frequency functions and gave numerical tables pertaining to it.† Fisher obtained as a special case a type of this frequency law for the frequency of the "intraclass" correlation coefficient.‡ Dodd investigated this probability function as a particular case when considering measurements under general laws of errors.§ The author obtained the law for the sum of n independent variables when each is subject to this hyperbolic law but was not able to express the sum function without the use of an integral.||

The object of this article is to find the probability function for the sum $\sum_{i=1}^{n} x_i$ when each variable x_i is subject to the probability function $(1/(2h)) \operatorname{sech} (\pi x_i/(2h))$, or to find the probability to within infinitesimals of higher order that

$$u \leqq \sum_{i=1}^{n} x_i \leqq u + du.$$

2. *Case* I: *n Finite.* If a general method due to Dodd¶ be applied to this hyperbolic secant law, the probability law for the sum of n independent variables is

* Presented to the Society, June 22, 1933.

† E. Roa, *A number of new generating functions with applications to statistics*, Thesis, University of Michigan, 1924.

‡ R. A. Fisher, *On the probable error of a coefficient of correlation deduced from a small sample*, Metron, vol. 1 (1920–21), pp. 3–32.

§ E. L. Dodd, *Functions of measurements under general laws of errors*, Skandinavisk Aktuarietidskrift, 1922, No. 3, pp. 134–158.

|| W. D. Baten, *Frequency laws for the sum of n variables which are subject to given frequency laws*, Metron, vol. 10 (1932), No. 3, pp. 75–91.

¶ E. L. Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, (2), vol. 27 (1925–26), p. 13.

$$p_n(u) = 2^n h^{-1}\pi^{-1}\int_0^\infty (e^x + e^{-x})^{-n} \cos (ux/h)dx.$$

The remainder of this section will be devoted to evaluating this definite integral for even and odd values of n. In order to make this evaluation clearer for any value of n let us first consider the case when n is equal to 4. The sum function is

$$p_4(u) = 2^4 h^{-1}\pi^{-1}\int_0^\infty (e^x + e^{-x})^{-4} \cos (ux/h)dx,$$

which can be found by integrating $F_4(z) = e^{iuz/h}(e^z + e^{-z})^{-4}$ around the contour C consisting of the following lines:

(a) the x-axis from $-R$ to $+R$, where R is large,

(b) the lines $z = \pm R + yi$,

(c) the line $z = \pi i + x$.

The only pole within the contour C is $(\pi i/2)$. By Cauchy's residue theorem, we have

$$(1)\quad \begin{aligned} &\frac{1}{2\pi i}\int_C (e^z + e^{-z})^{-4} e^{iuz/h}dz \\ &= \frac{1 - e^{-\pi u/h}}{2\pi i}\int_{-R}^{R} (e^x + e^{-x})^{-4} e^{iux/h}dx \\ &\quad + \frac{1}{2\pi i}\int_{\pi i}^{0} (e^{-R+yi} + e^{R-yi})^{-4} e^{iu(-R+yi)/h} i dy \\ &\quad + \frac{1}{2\pi i}\int_0^{\pi i} (e^{R+yi} + e^{-R-yi})^{-4} e^{iu(R+yi)/h} i dy \\ &= \text{the residue at } (\pi i/2), \end{aligned}$$

since the integral of $F_4(z)$ exists for h and u real quantities. The last two integrals approach zero as R becomes infinite, for

$$\lim_{R\to\infty} F_4(\pm R + yi) = 0.$$

The residue at $z = \pi i/2$ is also the coefficient of z^{-1} in the Laurent expansion of $F_4(z)$ around this point. Let $z = \pi i/2 + w$; then the residue of $F_4(z)$ at $\pi i/2$ is the residue of $F_4(\pi i/2 + w)$

at $w=0$. The function $F_4(\pi i/2+w)$ in the neighborhood of $w=0$ may be written in the form

$$\frac{e^{-\pi u/(2h)}}{(2w)^4}\, e^{iuw/h}\cdot g_4(w),$$

where

$$g_4(w) = \left(\frac{2w}{e^w - e^{-w}}\right)^4,$$

and, by definition, $g_4(0)=1/2$. The function $g_4(w)$ is analytic and can be expanded in a Maclaurin series in the neighborhood of the origin, hence

$$g_4(w) = g_4(0) + g_4'(0)w + g_4''(0)w^2/2 + g'''(0)w^3/3! + q_4(w),$$

where $q_4(w)$ is the remainder after the fourth term in the Maclaurin series representing $g_4(w)$. To find the coefficient of w^{-1} it is necessary to find the values of the first, second, and third derivatives of $g_4(w)$ at the point $w=0$. Newsom* obtained the following formula which will be used to find these derivatives at $w=0$:

$$\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^k\right]_{w=0} = \left[\frac{d^r}{dw^r}\left(\frac{2iw}{e^{iw} - e^{-iw}}\right)^k\right]_{w=0} = \frac{\sum \alpha_1\alpha_2\cdots\alpha_r}{{}_{k-1}C_r}, \tag{2}$$

in which k is any given positive integer $\geqq 2$, $1 \leqq r \leqq k-1$, and where $\sum\alpha_1\alpha_2\cdots\alpha_r$ denotes the sum of the $\binom{k-1}{r}$ products of r factors each formed by taking the possible combinations of the $(k-1)$ quantities $\pm(k-2)i$, $\pm(k-4)i, \cdots, \left\{\begin{smallmatrix}\pm i\\ 0\end{smallmatrix}\right\}$, r at a time; i having the usual interpretation, $i=(-1)^{1/2}$, and where $\left\{\begin{smallmatrix}\pm i\\ 0\end{smallmatrix}\right\}$ is understood as $\pm i$ or 0 according as k is odd or even.

Substituting $w=y/i$ in (2), we may write

$$\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^k\right]_{w=0} = \left[\frac{d^r}{dy^r}\left(\frac{2y}{e^y - e^{-y}}\right)^k\right]_{y=0}\cdot\left(\frac{1}{i}\right)^r,$$

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* C. V. Newsom, *On the derivatives of* $(w/\sin w)^k$ *at* $w=0$, American Mathematical Monthly, vol. 38 (1931), pp. 500–504.

from which the first three derivatives of $g_4(w)$ at $w=0$ can be found. Using these values, we find

$$g_4(w) = [1 - 4w^2/6 + q_4(w)];$$

hence

$$F_4(\pi i/2 + w) = \frac{e^{-\pi u/(2h)}}{2^2 w^2}\left(1 + \frac{iuw}{h} - \frac{u^2w^2}{2h^2} - \frac{iu^3w^3}{3!h^3} + \cdots\right)\left[1 - \frac{4w^2}{6} + q_4(w)\right];$$

and hence the coefficient of w^{-1} is found to be

$$\frac{-ie^{-\pi u/(2h)}}{2^4\cdot 3!h^3}u(u^2 + 4h^2).$$

By using this residue and by allowing R to become infinite in (1), we find that the probability law for the sum of four variables is

$$p_4(u) = \frac{u\cdot \text{csch}\,(\pi u/(2h))}{2\cdot 3!h^4}(u^2 + 4h^2).$$

The probability function

$$p_{2n}(u) = 2^{2n}h^{-1}\pi^{-1}\int_0^\infty (e^x + e^{-x})^{-2n}\cos\,(ux/h)dx$$

may be obtained in a similar way. To obtain this, it is necessary to find the coefficient of w^{-1} in the Laurent expansion of

$$F_{2n}(\pi i/2 + w) = \frac{e^{-\pi u/(2h)}e^{iuw/h}}{(2w)^{2n}}\left[\sum_{r=0}^{2n-1} g_{2n}^{(r)}(0)\frac{w^r}{r!} + q_{2n}(w)\right],$$

where

$$g_{2n}(w) = \left(\frac{2w}{e^w - e^{-w}}\right)^{2n},$$

and $g_{2n}^{(r)}(0)$ is the rth derivative of $g_{2n}(w)$ at 0, and $q_{2n}(w)$ is the remainder after the $2n$th term in the Maclaurin series representing $g_{2n}(w)$ in the neighborhood of $w=0$. According to Newsom's Theorem, we have

$$F_{2n}(\pi i/2 + w) = \frac{e^{-\pi u/(2h)}}{2^{2n}w^{2n}i^{2n}}\left[1 + \frac{iuw}{h} + \frac{(iuw)^2}{2!h^2} + \cdots + \frac{(iuw)^{2n-1}}{(2n-1)!h^{2n-1}} + \cdots\right]$$

$$\left[1 + \frac{\sum\alpha_1\alpha_2}{{}_{2n-1}C_2}\cdot\frac{w^2}{2} + \frac{\sum\alpha_1\alpha_2\alpha_3\alpha_4}{{}_{2n-1}C_4}\cdot\frac{w^4}{4!} + \cdots + \frac{\sum\alpha_1\alpha_2\cdots\alpha_{2n-2}}{{}_{2n-1}C_{2n-2}}\cdot\frac{w^{2n-2}}{(2n-2)!} + q_{2n}(w)\right],$$

from which the coefficient of w^{-1} is found to be

$$\frac{-iue^{-\pi u/(2h)}i^{2n}}{h^{2n-1}2^{2n}(2n-1)!}\left[u^{2n-2} + h^2\sum\alpha_1\alpha_2u^{2n-4} + h^4\sum\alpha_1\alpha_2\alpha_3\alpha_4u^{2n-6} + \cdots + h^{2n-2}\sum\alpha_1\alpha_2\alpha_3\cdots\alpha_{2n-2}\right].$$

The quantity in brackets is a polynomial in u^2 whose roots are equal to $-(2rh)^2$, where $r=1, 2, \cdots, n-1$. From this residue the probability function for the sum $u=\sum_{i=1}^{2n}x_i$ is found to be

$$p_{2n}(u) = \frac{u\cdot\operatorname{csch}(u\pi/(2h))}{2(2n-1)!h^{2n}}\prod_{r=1}^{n-1}\left[u^2 + (2rh)^2\right].$$

In a similar manner it can be shown that

$$p_{2n+1}(u) = \frac{\operatorname{sech}(\pi u/(2h))}{2\cdot h^{2n+1}(2n)!}\prod_{r=0}^{n-1}\left[u^2 + (2r+1)^2h^2\right].$$

3. *Case* II: *n Infinite.* By Liapounoff's theorem* the probability that

$$t_1(2B_n)^{1/2} < u < t_2(2B_n)^{1/2}$$

approaches

$$\pi^{-1/2}\int_{t_1}^{t_2} e^{-t^2}dt$$

* Liapounoff, *Sur une proposition de la théorie des probabilités*, Bulletin de L'Académie de St. Petersbourg, (5), vol. 13 (1900), pp. 358–386.

uniformly, where B_n is n times the second moment about the mean of the frequency distribution of the individual variable x, and t_1 and t_2 are any real numbers. The probability that

$$t_1(2B_n)^{1/2} < u < t_2(2B_n)^{1/2}$$

is

$$\int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} p_n(u)du,$$

and hence this expression approaches uniformly

$$\pi^{-1/2}\int_{t_1}^{t_2} e^{-t^2}\,dt$$

as n approaches infinity, or

$$\lim_{n\to\infty}\int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} p_n(u)du = \pi^{-1/2}\int_{t_1}^{t_2} e^{-t^2}\,dt,$$

or

$$\lim_{n\to\infty}\int_{t_1}^{t_2}(2B_n)^{1/2}p_n[(2B_n)^{1/2}u]du$$

$$= \lim_{n\to\infty} 2hn^{1/2}\int_{t_1}^{t_2} p_n(2hn^{1/2}u)du = \pi^{-1/2}\int_{t_1}^{t_2} e^{-t^2}\,dt,$$

since $(2B_n)^{1/2}=2hn^{1/2}$. Since the hyperbolic secant law, $(1/(2h))$ sech $(\pi x/(2h))$, is of bounded variation and the third moment of the absolute values of the chance variable x is finite, this function, or law, satisfies conditions mentioned by Cramér;* hence, according to Cramér's theorem,

$$2hn^{1/2}p_n(2hn^{1/2}u) \to \pi^{-1/2}e^{-u^2}.$$

On page 290 are plotted $2h\ 6^{1/2}p_6(2h\ 6^{1/2}u)$ and $\pi^{-1/2}e^{-u^2}$.

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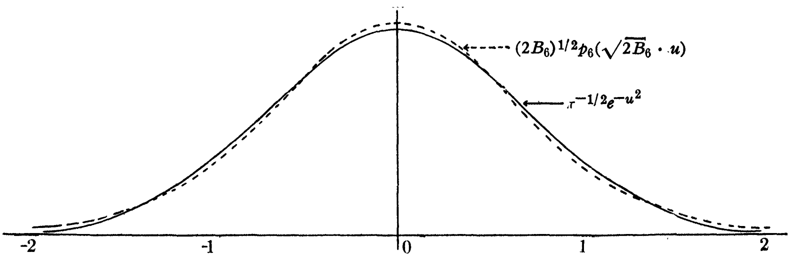
* Cramér, H., *On the composition of elementary errors, first paper; Mathematical deductions*, Skandinavisk Aktuarietidskrift, 1928, Nos. 1–2, p. 63.

4. *By-Products*. The function

$$(3)\qquad (2B_{2n})^{1/2}p_{2n}[(2B_{2n})^{1/2}\cdot u] = \frac{2nu \operatorname{csch}(n^{1/2}\pi u)}{(2n-1)!}\prod_{r=1}^{n-1}[4nu^2+(2r)^2].$$

Let n become very large and then substitute zero for u in (3). This should give a value near the value of $\pi^{-1/2}\cdot e^{-u^2}$ at $u=0$; hence



$$\frac{2^{2n}(n!)^2}{(2n)!n^{1/2}\pi}\to\frac{1}{\pi^{1/2}},\qquad \text{or}\qquad \frac{2^{2n}(n!)^2}{(2n)!(\pi n)^{1/2}}\to 1.$$

Dividing both numerator and denominator by $(2\cdot 4\cdot 6\cdot\ \cdots\ \cdot 2n)$ and squaring, we find

$$\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot\ \cdots\ \cdot(2n-2)(2n-2)2n}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot\ \cdots\ \cdot(2n-1)(2n-1)}\to\frac{\pi}{2},$$

which is a form similar to Wallace's formula.

When n is odd, a similar expression can be found which leads to this formula of Wallace.

If in $p_{2n}(u)$ and $p_{2n+1}(u)$, u be allowed to be zero, the following definite integrals are evaluated:

$$\int_0^\infty (e^{ht}+e^{-ht})^{-2n}\cos(ut)dt = \frac{1}{2^{2n+1}(2n-1)!\,h^{2n-1}}\prod_{r=1}^{n-1}(2rh)^2 = \frac{[(n-1)!]^2}{4(2n-1)!\,h},$$

$$\int_0^\infty (e^{ht}+e^{-ht})^{-2n-1}\cos(ut)dt = \frac{\Gamma[(2n+1)/2]}{\Gamma(n+1)\pi^{1/2}h}\cdot\frac{1}{2^{2n+1}}.$$

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