NOTE ON CUBIC SURFACES IN THE GALOIS FIELDS OF ORDER 2^n *

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Let us consider the cubic surface with the equation

$$
\begin{aligned}
f(x, y, z, w) \equiv a_0w^3 &+ (b_0x + b_1y + b_2z)w^2 + (c_0x^2 + c_1y^2 \\
&+ c_2z^2 + c_3yz + c_4zx + c_5xy)w + (d_0x^3 \\
&+ d_1y^3 + d_2z^3 + d_3y^2z + d_4yz^2 + d_5x^2y \\
&+ d_6xy^2 + d_7x^2z + d_8xz^2 + d_9xyz) = 0,
\end{aligned} \tag{1}
$$

whose coefficients and variables represent numbers in a Galois field of order 2^n. The first polar (or polar quadric) of any point $P'(x', y', z', w')$ with respect to (1) is

$$
\begin{aligned}
&(d_0x' + d_5y' + d_7z' + c_0w')x^2 + (d_6x' + d_1y' + d_3z' \\
&+ c_1w')y^2 + (d_8x' + d_4y' + d_2z' + c_2w')z^2 + (b_0x' + b_1y' \\
&+ b_2z' + a_0w')w^2 + (d_9z' + c_5w')xy + (d_9y' + c_4w')xz \\
&+ (c_5y' + c_4z')xw + (d_9x' + c_3w')yz + (c_5x' + c_3z')yw \\
&+ (c_4x' + c_3y')zw = 0.
\end{aligned} \tag{2}
$$

The second polar of P' with respect to (1) can be obtained from (2) by interchanging x' and x, y' and y, z' and z, w' and w. The polar quadric of (0, 0, 0, 1) is

$$c_0x^2 + c_1y^2 + c_2z^2 + a_0w^2 + c_3yz + c_4zx + c_5xy = 0. \tag{3}$$

The second polar of (0, 0, 0, 1) is

$$b_0x + b_1y + b_2z + a_0w = 0. \tag{4}$$

The Hessian of (1) is

$$
\begin{aligned}
(d_9z + c_5w)(c_4x + c_3y) + (d_9y + c_4w)(c_5x + c_3z)& \\
+ (c_5y + c_4z)(d_9x + c_3w) \equiv 0.&
\end{aligned} \tag{5}
$$

We note in fact, that the first polar of (0, 0, 0, 1) with respect to (3) vanishes identically; whereas the first polars of (1, 0, 0, 0), (0, 1, 0, 0), and (0, 0, 1, 0), respectively, are

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$$c_5 y + c_4 z = 0, \quad c_5 x + c_3 z = 0, \quad c_4 x + c_3 y = 0,$$

with

$$\begin{vmatrix} 0 & c_5 & c_4 \\ c_5 & 0 & c_3 \\ c_4 & c_3 & 0 \end{vmatrix} = 0,$$

all of which planes intersect in a line whose points have the coordinates (c_3, c_4, c_5, k) where k is arbitrary.

The statements in the preceding sentences are true only for the Galois fields of order 2^n. We shall now study the straight lines on the cubic surface (1) by a method that is valid also for cubic surfaces in any Galois field or in the ordinary real or complex three-dimensional space. The argument used is briefer than such a discussion as that in Salmon-Rogers, *Analytic Geometry of Three Dimensions*. To determine whether or not a line

$$(6) \qquad z = \lambda_3 x + \mu_3 y, \quad w = \lambda_4 x + \mu_4 y$$

lies on the surface (1), we substitute in (1) for x, y, z, w respectively the variables

$$(7) \quad y_1 = x + 0y, \quad y_2 = 0x + y, \quad y_3 = \lambda_3 x + \mu_3 y, \quad y_4 = \lambda_4 x + \mu_4 y.$$

We get the Taylor expansion

$$\begin{aligned}(8) \quad f(y_1, y_2, y_3, y_4) &= f(x + 0y, 0x + y, \lambda_3 x + \mu_3 y, \lambda_4 x + \mu_4 y) \\ &= x^3 f(1, 0, \lambda_3, \lambda_4) + x^2 y \left(0\frac{\partial'}{\partial y_1} + 1\frac{\partial'}{\partial y_2} + \mu_3 \frac{\partial'}{\partial y_3} + \mu_4 \frac{\partial'}{\partial y_4} \right) f \\ &\quad + \frac{xy^2}{2} \left(0\frac{\partial'}{\partial y_1} + 1\frac{\partial'}{\partial y_2} + \mu_3 \frac{\partial'}{\partial y_3} + \mu_4 \frac{\partial'}{\partial y_4} \right)^{(2)} f \\ &\quad + y^3 f(0, 1, \mu_3, \mu_4) = 0.\end{aligned}$$

By $\partial'/\partial y_i$ we mean a derivative with respect to y_i wherein the variables y_1, y_2, y_3, y_4 are subsequently replaced respectively by x, $0x$, $\lambda_3 x$ and $\lambda_4 x$. By the expression $(0\partial'/\partial y_1 + \cdots + \mu_4 \partial'/\partial y_4)^{(k)} f$ we mean the usual collection of terms containing kth derivatives of f with the factors in x and y removed. The number 2 in the denominator of $xy^2/2$ cancels the 2 in the coefficient.

If the line (6) lies entirely on (1) the coefficients of the terms in x^3, $x^2 y$, xy^2, and y^3 in (8) must vanish. Geometrically this means that the points $P_1(0, 1, \mu_3, \mu_4)$ and $P_2(1, 0, \lambda_3, \lambda_4)$ in which the line (6) cuts the planes $x=0$ and $y=0$ must both lie on (1), also P_1 must lie on the first and second polar of P_2 with respect to (1).

If we wish to find all such lines on (1) we must find the equation of condition ($K=0$) that a first degree equation in y, z, w (which is the line in which the second polar of $P'(x', 0, z', w')$ with respect to (1) cuts $x=0$) and a second degree equation in y, z, w (which is the conic in which the first polar of P' cuts $x=0$) and a third degree equation in y, z, w (which is the cubic curve in which (1) cuts $x=0$) shall have a common solution $P''(0, y'', z'', w'')$. Then we must solve $K=0$ (an equation in x', z', w') simultaneously with the equation obtained from (1) by putting therein $x=x'$, $y=0$, $z=z'$, $w=w'$. The result is too complicated to quote here. There turn out to be twenty-seven such lines.

To discuss the double points on (1) we use a method suitable only for the Galois fields of order 2^n. By suitable transformations we can reduce (1) to a form with $c_3=c_4=c_5=0$. We shall suppose this has been done. To find the condition for double points on (1) and the number of such double points, we take the first polars of (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) and find the condition for these polars to have common solutions and find also how many such solutions they can have. The desired equations are

$$\begin{aligned} b_0w^2 + d_0x^2 + d_6y^2 + d_8z^2 + d_9yz &= 0, \\ b_1w^2 + d_5x^2 + d_1y^2 + d_4z^2 + d_9xz &= 0, \\ b_2w^2 + d_7x^2 + d_3y^2 + d_2z^2 + d_9xy &= 0, \\ a_0w^2 + c_0x^2 + c_1y^2 + c_2z^2 &= 0. \end{aligned} \tag{9}$$

If $d_9 \neq 0$, we eliminate w^2 from the first three equations, using the fourth equation to do so. Then we square the first and third of the three resulting equations and use the second equation to eliminate y^2 from these two squared equations. We now have two equations in x and z of the form

$$ax^4 + bx^2z^2 + cxz^3 + dz^4 = 0, \quad \lambda x^4 + \mu x^3z + vx^2z^2 + \rho z^4 = 0. \tag{10}$$

From the form of the equations (10) we see that they can have only three, two, one, or no common roots in x/z. Hence in general (1) can have (if at least one of the terms in xyz, xzw, xyw, yzw is not missing) three, two, one, or no double points. All such double points will lie on the plane

$$a_0^{1/2}w + c_0^{1/2}x + c_1^{1/2}y + c_2^{1/2}z = 0. \tag{11}$$

We note as an exception to the above discussion of double points on (1) that if $a_0 = c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$, then we can use the first equation of (9) to eliminate w^2 from the next two equations and the two resulting quadratic equations in x, y, z may have four common solutions. Hence (1) may in this case have four double points.

If $d_9 = 0$, the equations (9) have just one common solution if and only if

$$\begin{vmatrix} d_0 & d_6 & d_8 & b_0 \\ d_5 & d_1 & d_4 & b_1 \\ d_7 & d_3 & d_2 & b_2 \\ c_0 & c_1 & c_2 & a_0 \end{vmatrix} = 0. \tag{12}$$

Therefore if $c_3 = c_4 = c_5 = d_9 = 0$ in (1) and if (12) is satisfied, then the cubic surface (1) has just one double point, otherwise it has no double points. We note that if and only if $c_3 = c_4 = c_5 = d_9 = 0$, every first polar (2) is a double plane.

If $a_0 = c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = d_9 = 0$, we see from the preceding paragraph that (1) has a line of double points. This holds also if for any other reason all the third-order minors of (12) vanish.

To find the equation of (1) in plane coordinates we assume that $c_3 = c_4 = c_5 = 0$ and proceed as follows. Solving $\delta w = \alpha x + \beta y + \gamma z$ simultaneously with (1), we obtain the equation

$$\begin{aligned} \lambda_0 x^3 + \lambda_1 y^3 + \lambda_2 z^3 + \lambda_3 x^2 y + \lambda_4 x y^2 + \lambda_5 x^2 z \\ + \lambda_6 x z^2 + \lambda_7 y^2 z + \lambda_8 y z^2 + d_9 xyz = 0. \end{aligned} \tag{13}$$

We note the presence of d_9 in (13). We note that

$$\lambda_0 = a_0\alpha'^3 + b_0\alpha'^2 + c_0\alpha' + d_0,\ \lambda_3 = a_0\alpha'^2\beta' + b_1\alpha'^2 + c_0\beta' + d_5,$$

where $\alpha' = \alpha/\delta$, $\beta' = \beta/\delta$, $\gamma' = \gamma/\delta$, and the other λ's have similar values in α', β', γ'. We seek now the equation of condition for the first polars of (1, 0, 0), (0, 1, 0), and (0, 0, 1) with respect to (13) to have a common solution. This equation of condition will be the equation of (1) in the plane coordinates α, β, γ, δ. The desired first polars are

$$(14)\qquad \lambda_0 x^2 + \lambda_4 y^2 + \lambda_6 z^2 = d_9 yz, \quad \lambda_3 x^2 + \lambda_1 y^2 + \lambda_8 z^2 = d_9 xz, \quad \lambda_5 x^2 + \lambda_7 y^2 + \lambda_2 z^2 = d_9 xy.$$

If $d_9 \neq 0$, we multiply together the first two equations of (14) and in the resulting equation replace $d_9 xy$ in the term $d_9^2 xyz^2$ by $\lambda_5 x^2 + \lambda_7 y^2 + \lambda_2 z^2$. Then we multiply together the first and third equations of (14) and in the resulting equation replace $d_9 xz$ in the term $d_9^2 xy^2 z$ by $\lambda_3 x^2 + \lambda_1 y^2 + \lambda_8 z^2$. Finally we multiply together the last two equations of (14) and in the resulting equation replace $d_9 yz$ in the term $d_9^2 x^2 yz$ by $\lambda_0 x^2 + \lambda_4 y^2 + \lambda_6 z^2$.

Then we take the square root of each of these three equations we have obtained and we get

$$(15)\qquad \begin{aligned} &(\lambda_0\lambda_3)^{1/2}x^2 + (\lambda_1\lambda_4)^{1/2}y^2 + (\lambda_8\lambda_6 + d_9\lambda_2)^{1/2}z^2 \\ &\qquad + \Lambda_2^{1/2}xy + (\Lambda_7 + d_9\lambda_5)^{1/2}xz + (\Lambda_5 + d_9\lambda_7)^{1/2}yz = 0, \\ &(\lambda_0\lambda_5)^{1/2}x^2 + (\lambda_4\lambda_7 + d_9\lambda_1)^{1/2}y^2 + (\lambda_6\lambda_2)^{1/2}z^2 \\ &\qquad + (\Lambda_8 + d_9\lambda_3)^{1/2}xy + \Lambda_1^{1/2}xz + (\Lambda_3 + d_9\lambda_8)^{1/2}yz = 0, \\ &(\lambda_3\lambda_5 + d_9\lambda_0)^{1/2}x^2 + (\lambda_1\lambda_7)^{1/2}y^2 + (\lambda_8\lambda_2)^{1/2}z^2 \\ &\qquad + (\Lambda_6 + d_9\lambda_4)^{1/2}xy + (\Lambda_4 + d_9\lambda_6)^{1/2}xz + \Lambda_0^{1/2}yz = 0, \end{aligned}$$

wherein Λ_i is the minor of λ_i in Δ_0, the determinant of the coefficients of x^2, y^2, z^2 in the three equations of (14).

If we look upon the six equations of (14) and (15) as linear equations in x_1, x_2, x_3, x_4, x_5, x_6, respectively, equal to x^2, y^2, z^2, yz, xz, xy, the equation of condition for them to have a common solution in these six variables is $\Delta = 0$, where Δ is the determinant of the coefficients of the six variables. The equation $\Delta = 0$ is at most of the twelfth degree in α, β, γ, δ.

If $d_9 = 0$, we see from (14) that the equation of (1) in plane coordinates is $\Delta_0 = 0$, which is at most of the ninth degree in α, β, γ, δ. We note that Δ reduces to Δ_0 for $d_9 = 0$.

From the above discussion we see that (1) has ordinarily an equation of the twelfth degree in plane coordinates, but that if all the terms in xyz, yzw, xyw, xzw are missing from (1), then the equation of (1) in plane coordinates is only of the ninth degree at most.

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