ON n-WEBS OF CURVES IN A PLANE

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This note contains a proof of Theorem 4 of the list given by W. Blaschke* in a preceding paper.

If $t_i = $ const. represents n sheaves of curves in a plane, then the maximal number of linearly independent relations

$$\sum_i U_{ik}(t_i) = 0, \quad (k = 1, \cdots, m, \; i = 1, \cdots, n), \tag{1}$$

is

$$N = \tfrac{1}{2}(n-1)(n-2). \tag{2}$$

Let (1) be any set of such relations; then we consider $U_{ik}(t_i)$, $(k=1, \cdots, m)$, for a fixed i to be the m coordinates of a point describing a curve $p_i(t_i)$ in an affine m-space.

If we can prove that the curves $p_i(t_i)$ all lie in parallel linear subspaces of dimension N, our theorem is proved, for this means that between the coordinates of every p_i there exist linear relations with the same constant coefficients, which express $m-N$ of the coordinates in terms of the other N. And this means that of the m relations (1) there can be only N linearly independent.

If we assume our functions U_{ik} to be differentiable a suitable number of times, however, this last statement comes down to proving that among the vectors

$$\frac{d}{dt_i} p_i(t_i) = p_i'(t_i), \quad p_i''(t_i), \quad p_i'''(t_i), \cdots, \tag{3}$$

there cannot be more than N linearly independent ones.†

We will prove this for $n=5$, $N=6$; the proof can easily be extended to all values of n. To avoid the use of many indices, we will write (1) in the form

$$p_1(u) + p_2(v) + p_3(r) + p_4(s) + p_5(t) = 0. \tag{4}$$

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* W. Blaschke, *Results and problems about n-webs of curves in a plane*, this Bulletin, vol. 38 (1932), p. 828.

† This does not really make it necessary to assume the functions (1) to be analytic; from a certain order m we can always replace (3) by an existence statement for solutions of a differential equation.

As our parameters t_i are given functions of the coordinates in the plane of our curves, and are all independent functions, we can express them as functions of u and v. We then differentiate the vector equation (3) with respect to u and v and find

$$(5)\quad \begin{aligned} 0 &= p_1' + p_3' r_u + p_4' s_u + p_5' t_u, \\ 0 &= p_2' + p_3' r_v + p_4' s_v + p_5' t_v; \end{aligned}$$

$$(6)\quad \begin{aligned} 0 &= p_1'' + p_3'' r_u^2 + p_4'' s_u^2 + p_5'' t_u^2 + p_3' r_{uu} + p_4' s_{uu} \\ &\quad + p_5' t_{uu}, \\ 0 &= p_2'' + p_3'' r_v^2 + p_4'' s_v^2 + p_5'' t_v^2 + p_3' r_{vv} + p_4' s_{vv} \\ &\quad + p_5' t_{vv}; \end{aligned}$$

$$(7)\quad \begin{aligned} 0 &= p_1''' + p_3'' r_u^3 + p_4'' s_u^3 + p_5''' t_u^3 + L(p_i'', p_i'), \\ 0 &= p_2''' + p_3''' r_v^3 + p_4''' s_v^3 + p_5''' t_v^3 + L(p_i'', p_i'); \end{aligned}$$

. . . .

$$(8)\quad 0 = p_3'' r_u r_v + p_4'' s_u s_v + p_5'' t_u t_v + p_3' r_{uv} + p_4' s_{uv} + p_5' t_{uv},$$

$$0 = p_3''' r_u^2 r_v + p_4''' s_u^2 s_v + p_5''' t_u^2 t_v + L(p_i'', p_i');$$

$$(9)\quad 0 = p_3''' r_u r_v^2 + p_4''' s_u s_v^2 + p_5''' t_u t_v^2 + L(p_i'', p_i'),$$

$$0 = p_3^{\text{iv}} r_u^3 r_v + p_4^{\text{iv}} s_u^3 s_v + p_5^{\text{iv}} t_u^3 t_v + L(p_i''', p_i'', p_i');$$

$$(10)\quad 0 = p_3^{\text{iv}} r_u^2 r_v^2 + p_4^{\text{iv}} s_u^2 s_v^2 + p_5^{\text{iv}} t_u^2 t_v^2 + L(p_i''', p_i'', p_i'),$$

$$0 = p_3^{\text{iv}} r_u r_v^2 + p_4^{\text{iv}} s_u s_v^3 + p_5^{\text{iv}} t_u t_v^3 + L(p_i''', p_i'', p_i');$$

. . . .

Here L always means a linear combination of the vectors in brackets, and $i = 3, 4, 5$. In this way we get two groups of equations. The first expresses all the derivatives of p_1 and p_2 as combinations of those of p_3, p_4, and p_5. The latter can be used to prove that of these there can be no more than 6 linearly independent.

If we assume for a moment that the equations (8), (9), (10) are not in a disturbing way dependent, then the result is obvious. For then we can have at the utmost 3 independent vectors p_i', of the vectors p_i'' one can be expressed by means of the others and p_i' as a consequence of (8), so we get only two extra independent vectors, and (9) shows that vectors p_i''' can give only one extra dimension. The total number is exactly $3+2+1=6$.

So the only thing that remains to be proved is that relations (8), (9), (10) are really independent. Now in (8) the coefficients of p_i'' cannot vanish. For $r_u = 0$ would mean that r was a function of u alone, and therefore that sheaves $r = \text{const.}$ and $u = \text{const.}$ would coincide. So (8) really gives us a relation between the p_i''. To show that (9) gives 2 relations we have to consider the matrix

$$\begin{Vmatrix} r_u^2 r_v & s_u^2 s_v & t_u^2 t_v \\ r_u r_v^2 & s_u s_v^2 & t_u t_v^2 \end{Vmatrix}$$

and show that it is of rank 2. But one of the determinants is

$$r_u r_v s_u s_v \cdot \begin{vmatrix} r_u & s_u \\ r_v & s_v \end{vmatrix}$$

and none of the factors can vanish, the last one since this would imply the dependence of the functions r and s, which is again impossible. Finally the essential determinant in (10) is equal to

$$r_u r_v s_u s_v t_u t_v \begin{vmatrix} r_u & s_u \\ r_v & s_v \end{vmatrix} \begin{vmatrix} s_u & t_u \\ s_v & t_v \end{vmatrix} \begin{vmatrix} r_u & t_u \\ r_v & t_v \end{vmatrix}$$

so that from (10) we can really compute p_i^{iv} as linear combinations of p_i''', p_i'', p_i'. We see that there is no danger for dependency of the equations, and our theorem is proved.

Of course if $n > 5$, we have a similar proof, only the determinants we have to consider are of higher order. We find

$$N = n - 2 + n - 3 + n - 4 + \cdots + 2 + 1 = \tfrac{1}{2}(n-1)(n-2).$$

As a corollary, for $n = 4$, we have: *If a 2-dimensional surface in k-space can be generated in two different ways as a translation surface, it lies in a linear three-dimensional subspace.**

For the assumption leads to a vector equation (4) with $n = 4$ and our formula gives $N = 3$.

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* See S. Lie, Leipziger Berichte, 1897, p. 186.