SOLUTION OF THE ZARANKIEWICZ PROBLEM*

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1. *Introduction.* In 1925 C. Zarankiewicz† proposed the following problem: *Is every acyclic continuous curve‡ homeomorphic with some proper subset of itself?* It is the purpose of this paper to show that the above question is to be answered in the negative.

Our result will depend upon the following theorem.

THEOREM. *The acyclic continuous curve S is homeomorphic with no proper subset of itself if it contains a set K such that* (1) *each point of K is a fixed point with respect to any* (1, 1) *bicontinuous transformation of S into a subset of itself*; *and* (2) *each point of S of* (*Urysohn-Menger*) *order* >1 *lies on an arc of S whose end points are points of K.*

PROOF. Let p be any point of S of order >1. There is an arc a_1a_2 in S which contains p and whose end points are points of K. Let T be any (1, 1) bicontinuous transformation of S into a subset of itself. Since $T(a_1)=a_1$ and $T(a_2)=a_2$, and since there is just one arc in S from a_1 to a_2, T must carry a_1a_2 into itself. Hence there is a point q of a_1a_2 such that $T(q)=p$. Thus the subset of S into which T carries S must contain all points of S of order >1. As these points are dense in S, this subset must be S itself.

Our problem, then, is to construct an acyclic continuous curve which satisfies the conditions of the above theorem. We shall first define certain auxiliary sets $E_{x_1x_2\cdots x_k}$.

2. *Definition of the Sets* $E_{x_1x_2\cdots x_k}$. Within a linear interval ab choose points a_n so that $a_{n+1}<a_n$ and $\lim a_n=a$. Within each interval $a_{n+1}a_n$ choose points $a_{n,m}$ so that $a_{n,m}<a_{n,m+1}$ and $\lim_{m=\infty} a_{n,m}=a_n$. At each point a_n and $a_{n,m}$ erect a perpendicular to ab. Take these perpendiculars so that for any $\epsilon>0$ only a finite number of them have a length $>\epsilon$. The set of points obtained in this way will be called a set E_1. The point a will be

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† Fundamenta Mathematicae, vol. 7, p. 381, problem 37.

‡ The term *continuous curve* is used throughout the present article to mean a compact, locally connected, metric continuum.

called the *origin* of E_1 and the perpendiculars which we have erected will all be referred to as *perpendiculars of rank* 1. It is clear that E_1 is an acyclic continuous curve. Everything will be the same in the definition of E_2 except for this one change: the points $a_{n,m}$ are taken within $a_{n+1}a_n$ so that $a_{n,m+1} < a_{n,m}$ and $\lim_{m=\infty} a_{n,m} = a_{n+1}$.

Let us now suppose that we have defined sets $E_{x_1x_2\cdots x_i}$, where x_i $(i \leqq k)$ can have either of the two values 1 and 2. Let us suppose furthermore that we have defined the expressions *origin of* $E_{x_1x_2\cdots x_k}$ and *perpendiculars of rank k of* $E_{x_1x_2\cdots x_k}$. We suppose finally that $E_{x_1x_2\cdots x_k}$ has been so defined that it is an acyclic continuous curve. To define the set $E_{x_1x_2\cdots x_k1}$, we proceed as follows. We replace each perpendicular of rank k of $E_{x_1x_2\cdots x_k}$ by a set E_1 whose origin is the foot of that perpendicular. Furthermore we do this, as we clearly can, so that the resulting set $E_{x_1x_2\cdots x_k1}$ is an acyclic continuous curve. By *origin of* $E_{x_1x_2\cdots x_k1}$ we will mean merely the origin of $E_{x_1x_2\cdots x_k}$, and by *perpendiculars of rank* $k+1$ *of* $E_{x_1x_2\cdots x_k1}$ the perpendiculars of rank 1 of the sets E_1 employed in obtaining $E_{x_1x_2\cdots x_k1}$ from $E_{x_1x_2\cdots x_k}$. Everything will be the same in the definition of $E_{x_1x_2\cdots x_k2}$ except for this one change: in obtaining $E_{x_1x_2\cdots x_k2}$ from $E_{x_1x_2\cdots x_k}$ we shall employ sets E_2 instead of sets E_1.

3. *Construction of an Acyclic Continuous Curve which Satisfies the Conditions of the Theorem.* This construction will be achieved through the use of the following sequence of sets:

$$E_1, E_{21}, E_{221}, \cdots, E_{22\cdots21}, \cdots.$$

Let us first re-label these sets in the order named as

$$W_1, W_2, W_3, \cdots, W_n, \cdots.$$

We begin with a set W_1 whose origin is a point a and adjoin to it three line segments ac, ad, and ae so that the only point which any two of the sets W_1, ac, ad, and ae have in common is the point a. Let us denote the resulting acyclic continuous curve by S_1. We now consider the arc in S_1 from each end point of S_1 to a. There will be a first branch point of S_1 in the order from the end point of S_1 to a on such an arc, and the portion of the arc from the end point of S_1 to this branch point is a line segment. Denote the mid-point of this segment by x. We obtain, of course, a countable infinity of points x. With this countable infinity of points we associate in a (1, 1) way* the sets of odd index

$$W_3, W_5, W_7, \cdots$$

and take x as the origin of the associated set $W(x)$ in such a way that S_1 and $W(x)$ have only the point x in common. Also we attach to the point x a straight line segment having x as one end point and having only x in common with $W(x)+S_1$. All this can clearly be done so that the resulting set S_2 is an acyclic continuous curve. Now S_3 will be related to S_2 in the same way as S_2 is related to S_1, except that we make use of sets $W_{2(2m+1)}$ instead of sets W_{2m+1}. In general S_{n+1} is related to S_n in the same way as S_n is related to S_{n-1}, except that we make use of sets $W_{2^{n-1}(2m+1)}$ instead of sets $W_{2^{n-2}(2m+1)}$. Now constructions of the general type just described are common in the literature and it is well known† that such a construction can be carried through so that the closure of the sum of the acyclic continuous curves successively obtained is itself an acyclic continuous curve. We may suppose then that $S=(\overline{\sum_{n=1}^{\infty}S_n})$ is an acyclic continuous curve.

It will now be shown that S satisfies the conditions of our theorem. We notice first that any branch point of S is either of order 3 or order 4. The points of order 4 are the point a of S_1 and the points x which arise at successive stages of our process of construction. We will denote the set of points of order 4 of S by K, and it will be shown that K has the properties of the set K in our theorem. In fact, it is obvious from the way in which S was constructed that K has property (2) of the theorem. We need only show that it has property (1).

In the first place, we notice that if T is any (1, 1) bicontinuous transformation of S into a subset of itself, T must carry each point of K into a point of K, since no point of S is of order >4 and K contains all points of order 4 of S. Let us suppose that there are two *distinct* points q_1 and q_2 of K such that $T(q_1)=q_2$. Let us suppose for definiteness (the argument is similar in the opposite case) that the set W which has q_1 as its origin is of

* It is clear that this (1, 1) correspondence can be made perfectly definite.

† For a similar construction and proof that the result is an acyclic continuous curve see K. Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

lower index than the set W which has q_2 as its origin. If we consider any point q of K we notice that of the four essentially distinct arcs of S which meet in q just one has the property that the branch points on it have q as limit point. Let us denote this arc by qb and take the point b so close to q that qb is a line segment and contains no point of order 4 other than q. Let us now consider the arcs q_1b_1 and q_2b_2. It is clear that there is a sub-arc q_1b_1' of q_1b_1 and a sub-arc q_2b_2' of q_2b_2 such that the transform of q_1b_1' is q_2b_2'. Any branch point of S on q_1b_1' is transformed into a branch point of S on q_2b_2'. If $W(q_1) = W_1$, we see that we have already reached a contradiction. For $W_1 = E_1$ and $W(q_2) = W_m = E_{2\cdots 1}$, which means that q_1b_1' contains branch points which are limit points of branch points from *the left*, while q_2b_2' contains no such points. If $W(q_1) = W_2$, we fix our attention upon some one branch point of S interior to q_1b_1'. Let us denote this point by r_1 and the corresponding point on q_2b_2' by r_2, and consider the perpendiculars to q_1b_1' and q_2b_2' at r_1 and r_2, respectively. Denote these perpendiculars by r_1s_1 and r_2s_2. Now since $W_2 = E_{21}$, r_1 is a limit point along r_1s_1 of branch points of S which are in turn limit points of branch points of S from *below* along r_1s_1, while r_2s_2 contains no such points since $W(q_2) = W_m = E_{22\cdots 1}$. It is obvious that the argument exemplified above can be extended to apply to the general case where $W(q_1) = W_n$ and $W(q_2) = W_m$ whether $n < m$ or $m < n$. It follows that $q_1 = q_2$ if q_1 and q_2 belong to K and $T(q_1) = q_2$.

In conclusion it may be remarked that it is possible to construct an acyclic continuous curve which contains no point of order >3 and which is homeomorphic with no proper subset of itself. We need only employ sets E_{212}, E_{2112}, E_{21112}, $\cdots$ instead of sets E_1, E_{21}, E_{221}, $\cdots$. The proof will involve only a few more details than the proof given here.

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