NOTE ON TRIPLE SYSTEMS*

BY R. D. CARMICHAEL

A Steiner triple system is said to have the property E if (and only if) the presence of the triples (abu), (bcv), (ucx) implies the presence of the triple (avx) in the system. For such a triple system Th. Skolem† has proved in an elementary way that there exists such a system when and only when the number n of elements involved is of the form $2^{k+1}-1$, where k is a positive integer and that for every n of such form there exists but one triple system with the property E. H. Hasse,† by means of Abelian groups, has given a simple proof of this theorem of Skolem.

Since a Steiner triple system formed from the n elements $x_1, x_2, \cdots, x_n$ has the property that each pair x_ix_j of elements in the set belongs to one and just one triple of the set forming the triple system and since this system is now to be restricted so as to have the property E it follows that the elements $x_1, \cdots, x_n$ and the triples of the system may be taken to be the points and the lines respectively of a finite geometry in the sense of Veblen and Bussey,‡ as one may readily see by comparing the properties of the system with the postulates of the geometry. The property E of the system is equivalent to the transversal property in the geometry.

Since the geometry contains just three points on a line it follows from the general theorem of Veblen and Bussey concerning the existence of finite geometries that this geometry is precisely their $PG(k, 2)$ consisting of $2^{k+1}-1$ points arranged in lines of three points each. These triples of points, constituting the lines of the geometry, form the required triple system with the property E and the system is unique. Thus it follows that the theorem of Skolem is essentially a special case of the existence theorem for the finite geometries.

---

* Presented to the Society, September 2, 1932.

† Th. Skolem, Norsk Matematisk Tidsskrift, vol. 13 (1931), pp. 41–51; H. Hasse, ibid., vol. 13 (1931), pp. 105–107.

‡ Veblen and Bussey, Transactions of this Society, vol. 7 (1906), pp. 241–259.

The connection of these triple systems with Abelian groups of order 2^k and type $(1, 1, \cdots, 1)$, brought out by Hasse (loc. cit.), is a special case of a general result which I have established.*

The largest permutation group on $x_1, x_2, \cdots, x_n$ each element of which transforms the triple system into itself is said to be a *triple* group and to be the group *belonging* to the triple system. From the theory of the finite geometries $PG(k, 2)$ it follows at once that the triple group G belonging to the triple system of Skolem is the collineation group in $PG(k, 2)$. When $k>1$ the group G is doubly transitive. (It is obvious that no triple group of degree greater than 3 can be triply transitive.) Moreover, the group G contains a cyclic permutation on all its elements, such a permutation arising readily from a primitive mark of the Galois field $GF[2^k]$. By means of such a cyclic permutation in G any particular triple in the system is transformed successively into all the triples of the system.

The euclidean finite geometry $EG(k, 3)$, in the notation of Veblen and Bussey, also affords an example of a triple system, the lines in the geometry constituting the triples in the system. The elements from which this triple system is formed are 3^k in number, where k is a positive integer. The triple group belonging to the system is the collineation group in the geometry; it is doubly transitive when $k>1$.

THE UNIVERSITY OF ILLINOIS

* R. D. Carmichael, American Journal of Mathematics, vol. 52 (1930), pp. 745–788; see pp. 746 ff.