ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Professor W. L. Ayres: *On a certain neighborhood property.*

The paper considers metric spaces in which the frontier of every proper neighborhood contains at least one ordinary point. An open subset D such that the space contains a point not belonging to $\overline{D}$ is a proper neighborhood. After an examination of the properties of such spaces, the results are applied to obtain new characterizations of the arc and of the simple closed curve. (Received November 7, 1931.)

2. Mr. H. P. Thielman: *An integral addition theorem for Bessel functions.*

In this paper a new integral addition theorem for Bessel functions is derived. The results are obtained by means of a Volterra transformation and its inverse. (Received November 17, 1931.)

3. Dr. Gordon Pall: *On the application of a theta formula to representation in binary quadratic forms.*

It is shown that a certain method of Nazimoff's leads to the number of representations in ax^2+by^2, ab odd, only in the cases $a=1$, $b=1, 3, 5, 7$. Details are supplied for the case $a=1$, $b=7$. (Received November 23, 1931.)

4. Professor H. H. Germond: *A modification of Stirling's formula arising from an asymptotic evaluation of the Wallis integral.*

One method of deriving Stirling's approximation formula is to seek a function, y, whose derivative for each positive integer n is equal to the average rate of change of $n!$ in passing from $n-1$ to $n+1$. The asymptotic expression thus obtained is $y=C(n/e)^n(n+1)^{1/2}$. We determine C as follows. The integral $W(n)=\int_0^{\pi/2}\sin^n\theta d\theta$ has the value $(n!\pi/2)2^n(n/2)!^2$, when n is a positive even integer, and the value $(2^n[(n-1)/2]!^2/2)n!$ when n is a positive odd integer. Taking the geometric mean of $W(n)$ and $W(n+1)$ as a first approximation to the value of $W(n+1/2)$, we obtain a general asymptotic expression $W_1(n)=(\pi/(2n+1))^{1/2}$ for $n>0$. Replacing the factorials in the correct expression for W by the corresponding approximate expressions, and equating W to W_1 leads to a determination of C such that $y=(n/e)^n(2\pi(n+1/6))^{1/2}$. The values of y thus computed agree with the values of $n!$ to within one-half of one per cent, even when $n=1$. By further steps, a simple interpolation formula is derived which expresses $\Gamma(n+\epsilon)$ in terms of $\Gamma(n)$. (Received November 28, 1931.)

5. Professor J. B. Coleman: *Another coefficient of linear correlation between two variables.*

The method of least squares is applied to a single line of best fit to two variables. A coefficient is obtained, invariant with respect to change of units on the axes, as well as with respect to rotation of the axes. The coefficient reduces to a simple function of the Pearson product-moment coefficient, $1-((1-|r|)/(1+|r|))^{1/2}$, in which r is the Pearson coefficient. (Received November 19, 1931.)

6. Mr. C. H. Fischer: *On certain non-normal correlation surfaces.*

The present paper is concerned with the correlation surface $u=F(x, y)$, where the variables x and y are, respectively, the sum of m and of n independent values of a variable t, and have k values of t in common. The variable t is characterized by a rather arbitrary continuous distribution function, $f(t)$. The general equation of the correlation surface is expressed in terms of iterated integrals. The regression of each variable on the other is linear, and the equations of the regression lines involve only a single integral. The coefficient of correlation between x and y, which is found to be independent of the distribution function, $f(t)$, is $k/(mn)^{1/2}$. These results are extended to more than two variables, where each variable, x_i, after the first, consists of the sum of some k_i constituent values of t selected at random from the preceding variable, x_{i-1}, plus the sum of some (n_i-k_i) values of t, each selected at random directly from the parent distribution function $f(t)$. (Received November 23, 1931.)

7. Professor H. L. Rietz: *On the Lexis theory and the analysis of variance.*

In a paper published in 1921 on a generalization of the Lexis theory for the classification of statistical series with reference to their dispersion into Bernoulli, Poisson, and Lexis types, Coolidge based much of his reasoning on the fundamental dispersion theorem of n variates. In setting up criteria for the practical classification of actual statistical series, Coolidge followed the customary procedure of introducing approximations by replacing $(n-1)/n$ by unity. While certain estimates of population variance given by R. A. Fisher seem to have been obtained as inferences based largely on the number of degrees of freedom of the variates rather than by even moderately rigorous formal mathematical proofs, it is first shown in the present paper that the reasoning of Coolidge leads to exactly the same estimates if we avoid the approximation to which reference is made above. The paper next considers the appropriateness of different estimates of variance for particular purposes in the analysis of the inner structure of data classified into rows and columns. (Received November 16, 1931.)

8. Dr. A. T. Craig: *The simultaneous distribution of mean and standard deviation in small samples.*

The simultaneous distribution function of the arithmetic mean $\bar{x}$ and standard deviation s of samples of n items drawn from a normal universe is known, and is of the form $F(\bar{x}, s) = \phi(\bar{x})\psi(s)$. In the present paper, several theorems are established which make possible, in case of small samples, the determination of $F(\bar{x}, s)$ when the parent population, characterized by a continuous frequency function $f(x)$, is of a rather arbitrary type. The correlation surfaces thus obtained possess some interesting properties. (Received November 24, 1931.)

9. Professor H. H. Germond: *A closed expression approximating to the value of the probability integral.*

The solution of certain problems might be aided, could the value of the probability integral, $I = (2/\pi^{1/2})\int_0^z e^{-x^2}dx$, be expressed in closed terms. I^2 may be viewed as the integral of $(4/\pi)e^{-r^2}$ (where $r^2 = x^2 + y^2$) taken over the square bounded by the x and y axes and the lines $x = z$ and $y = z$. Let P^2 be the integral of the same function, but taken over the quadrant of a circle of the same area as the square and with its center at the origin. Then $P = (1 - e^{-4z^2/\pi})^{1/2}$. The values of P are only slightly greater than the corresponding values of I, the maximum difference being less than three-quarters of one per cent, and occurring in the neighborhood of $z = 1.1$. For values of z greater or less than this, the two functions converge. The difference between the two functions may be written in the form $az^3e^{-bz^2} + \cdots$. (Received November 28, 1931.)

10. Dr. A. T. Craig: *On the correlation between certain averages.*

This paper is concerned with the correlation functions of certain averages of a statistical variate x which obeys a given continuous law of probability $f(x)$. The averages considered in the present paper are the arithmetic mean, the median, and the range. The correlation functions of the mean and median and of the mean and range are given for small samples; that of the median and range is given for samples of arbitrary size. (Received November 24, 1931.)

11. Professor E. L. Dodd: *Frequency laws showing stability with reference to the geometric mean and other means.*

A large class of means may be constructed with the aid of a continuous increasing function $y = \xi(x)$ and its inverse $x = \xi^{-1}(y)$ by taking for the mean, X, of n variates $x_1, x_2, \cdots, x_n$, $X = \xi^{-1}[(1/n)\Sigma_1^n \xi(x_i)]$. It is then possible to carry over to these means much that is known regarding stability with respect to the arithmetic mean. If the logarithm of the characteristic function, in the sense of Levy, is $-c|t|^\alpha$, with $0 < \alpha \leqq 2$, and if $\xi(x)$ is an odd function, then from Probability $\{|x_i| < v\} = F[\xi(v), \alpha, c]$, $i = 1, 2, \cdots, n$, we can conclude that Probability $\{|X| < v\} = F[n^{1-1/\alpha}\xi(v), \alpha, c]$. To obtain a geometric mean, G, we take $\xi(x) = \log x$. Then if a is a positive constant, and if Probability $\{a/v < x_i < av\} = F[\log v, \alpha, c]$, $v \geqq 1$, it follows that Probability $\{a/v < G < av\} = F[n^{1-1/\alpha} \log v, \alpha, c]$. (Received November 28, 1931.)

12. Dr. S. S. Wilks (National Research Fellow): *Distributions of certain statistics in matched pairs of samples from a normal bi-variate population.*

The object of this paper is to investigate the distributions of population parameters estimated by the method of maximum likelihood for a frequently used type of sampling of the following description. A sample of N items is drawn at random from a normal population of two variables x and y, followed by a second sample of N items restricted only in that its distribution of x's shall be identical, item by item, with that of the other. The joint distribution and moments of the estimated variances and covariance were found by means of Romanovsky's integral equation method. Two different estimates can be obtained for the variance of the difference between the two sets of y's; the one from the two sets of y's alone and the other from the estimated variances and covariance of the population of x and y. The latter estimate is found to be distributed with smaller variance than the former. Similar results were found in the distribution of the two estimates of "Student's" ratio. (Received November 20, 1931.)

13. Dr. S. S. Wilks (National Research Fellow): *Determination of characteristic functions of quadratic statistics by determinants.*

It has been shown by Karl Pearson (Royal Society Philosophical Transactions, (A), vol. 200 (1903)) and others that $\int_{-\infty}^{\infty} e^{-\theta} dV = \pi^{n/2}\Delta^{-1/2}$, where θ is a positive definite quadratic form in n variables, Δ the determinant of the symmetric matrix of θ, and dV the product of the n differentials, the integral being taken over the entire n-dimensional space. By definition, the characteristic function of the quadratic form w for a sample of n items from a normal population is proportional to $\int_{-\infty}^{\infty} e^{-\theta+\alpha w} dV$, which, for sufficiently small values of α, can be evaluated from the determinant of the matrix of $(\theta-\alpha w)$. The extension of this method to any finite number of quadratic statistics in any finite number of variates in any finite sample is immediate. It is also applicable to linear forms and combinations of the two types of statistics. The determinants arising in connection with variances and covariances are of a form that proved to be easily evaluated, thus affording a comparatively simple determination of the characteristic functions reached by Romanovsky (Metron, vol. 5 (1925), No. 4, and vol. 7 (1927), No. 3) by direct integration. (Received November 20, 1931.)

14. Mr. Albert Wertheimer: *A generalized error function.*

Given a set of numbers l_i, $i=1, 2, \cdots, n$, obtained by making n observations under the same conditions on a single quantity, l. We wish to define a certain pair of functions $f(p, l_i)$ and $\phi(p, l_i)$ such that the equation $f(p, l_i)=0$ defines p as a unique value assigned to the observed quantity l, and $\phi(p, l_i)dl_i$ gives to within infinitesimals of higher order the probability that the observed value will lie in the interval bounded by l_i and l_i+dl_i. In this paper, we assume that $\Sigma(\partial/\partial p)\log\phi(p, l_i)=0$ and $f(p, l_i)\equiv p-F(l_1, l_2, \cdots, l_n)$ where F is continuous and single-valued with continuous derivatives up to the second order. We also assume that F is a symmetric function, $F_{l_i}=F_{l_i}(F, l_i)$, and $F_{l_i}\neq 0$ in the region considered. A general error function is then derived from which the so-called normal error law, the Poincaré function, and many other forms of the error function as well as the Pearson curves are derived as special cases. We also show that there exists a certain function $g(p, l_i)$ uniquely determined from F, and if the assumption is made that $\phi(p, l_i) \equiv \phi\{g(p, l_i)\}$, we obtain as a special case a generalized normal error function, namely $\phi\{g\} = ce^{-h^2g^2}$. (Received November 17, 1931.)

15. Professor C. G. Latimer: *On the class number of an Abelian field.*

Let F be an algebraic field of odd prime degree e, and a primary divisor of the field defined by a primitive Mth root of unity. If M is a prime, it is known that the class number of F is $h = \pm\delta/R$, where R is the regulator of F and δ is a determinant obtained from R by replacing the fundamental units by a unit η of F and $e-2$ of its conjugates (Fueter, Journal für Mathematik, vol. 147, p. 183). More generally, h may be written in this form whenever M is prime to e. In this paper, we show that if h may be so written, then it is the norm of an ideal in the field defined by a primitive eth root of unity. Eisenstein obtained an equivalent result for the case $e=3$, M a prime (Journal für Mathematik, vol. 29, p. 49). We also show that a necessary and sufficient condition that F contain a unit η such that η and $e-2$ of its conjugates form a fundamental set is that the above mentioned ideal be principal. (Received November 27, 1931.)

16. Professor B. W. Jones: *On the reduction of positive quaternary quadratic forms.* Preliminary communication.

L. Charve (Annales Scientifiques de l'Ecole Normale Supérieure, (2), vol. 11(1882), pp. 119–134), seeking to define a reduced positive quaternary quadratic form, uses a method similar to that which he used for ternaries. He associates with f the form $\phi = a(x-y)^2+b(x-z)^2+c(x-t)^2+d(x-u)^2+e(y-z)^2+f(y-t)^2+g(y-u)^2+h(z-t)^2+k(z-u)^2+l(t-u)^2$. He calls f reduced when (roughly) not more than two of the coefficients of ϕ are negative. For example, when none are less than zero he says that all ϕ equivalent to a given ϕ are identical except for permutations of *variables*. They are in reality identical except for a permutation of *coefficients* $a, b, \cdots$, which for such a form is quite a different thing. This and more important inaccuracies are corrected in this paper, and progress is made toward defining a unique reduced form f. (Received November 24, 1931.)

17. Dr. L. A. Dye and Professor F. R. Sharpe: *The quartic space involutorial transformations with a double conic.*

A conic C_2, a rational quartic C_4, meeting C_2 in 4 points and a point O, determine a web of quartic surfaces having C_2 as double curve and passing through C_4 and O, so that Cremona transformations of this type exist. They may be regarded as the product of the quadratic inversion C_2, $O \equiv S$, and a cubic transformation T having C_2 and C_4 as a fundamental composite sextic curve of genus 3, so that T can be defined by 3 bilinear equations. In this paper the conditions are found that ST shall be involutorial, and also the actual equations of the transformations when these conditions are satisfied. (Received November 18, 1931.)

18. Dr. M. M. Slotnick: *On fundamental transformations of surfaces.*

By means of the analytic method for the study of the projective differential geometry of conjugate nets discussed in a recent paper by the author (American Journal of Mathematics, vol. 53, No. 1, Jan., 1931) which defines an arbitrary harmonic congruence to the net and its dual, an arbitrary conjugate congruence to the net, the general transformation F can be set up. This method, because of the symmetry of the parameters and other advantages, offers another means of studying the properties of these transformations. (Received November 23, 1931.)

19. Professor L. R. Ford: *Isometric properties of projective transformations.* Preliminary communication.

A projection involving two independent complex variables may be considered as a one-to-one transformation of the points of a four-dimensional real space. The locus of a point in the neighborhood of which volumes are unaltered is a hypercylinder, which will be called the *isometric* hypercylinder. Volumes within this hypercylinder are magnified, those without are diminished. The special connections of this hypercylinder with the geometry of the projection are being studied. For example, the product of the distances of the three fixed points from its axis is equal to the cube of its radius. It is hoped that the isometric hypercylinder will be of use in the study of properly discontinuous groups of projections. (Received November 28, 1931.)

20. Professor Nathan Altshiller-Court: *On the isodynamic points of four spheres.*

The isodynamic points W, W' of four spheres (A), (B), (C), (D) are, by definition, the points common to the six spheres of similitude of the given spheres taken in pairs. The four inverses of $W(W')$ with respect to (A), (B), (C), (D) determine a sphere (R') concentric with the orthogonal sphere (R) of the given spheres, and W is a center of similitude of (R') and the sphere (O) determined by the centers A, B, C, D of the four given spheres. If O, R are the centers, and r, s the radii of the spheres (O), (R), we have $s^2=(1-k)\cdot(r^2-k\cdot OW^2)$, where $k=OR:OW$. These properties are readily applicable to a tetrahedron, for the centroid G of a tetrahedron $ABCD$ is an isodynamic point of the four spheres having their radii proportional to the medians of $ABCD$ and for centers points dividing these medians in the same ratio. As a further application the variable tetrahedron is considered which is inscribed in a fixed sphere and has a fixed centroid. The spheres having for diameters the medians of this tetrahedron are orthogonal to a fixed sphere having for its center the Monge point of the tetrahedron. (Received November 28, 1931.)

21. Professor C. O. Oakley: *Semi-linear equations. Part* II: *Irregular polygons.*

The author continues the study of semi-linear equations and their geometric interpretations. A semi-linear equation is of the form $u_0+m_1|u_1|+\cdots+m_n\cdot|u_n|=0$, where $u_i=a_ix+b_iy+c_i$, $i=0, 1, \cdots, n$; it is said to be of order

$p(\leqq n)$ when there are exactly p non-vanishing coefficients among the m's. Problems of the representation of certain broken-line curves, polygons, polygonal areas, etc., by means of semi-linear equations of restricted order are treated. For equations of small order, the inverse problem is also treated. (See abstract 179 in this Bulletin, May, 1931, for Part I: Regular polygons.) (Received November 28, 1931.)

22. Professor H. C. Shaub: *Plane Cremona transformations arising from a generalization of the transformation by reciprocal radii.*

The generalizations treated in this paper arise from an equation of the form $r\, r_1 = R(x, y)$, where R is a rational function and r, r_1 the radii vectores of a point P and its image P_1. The transformations are of the perspective deJonquières type. Certain modifications, giving rise to involutions, are also considered. (Received December 2, 1931.)

23. Professor A. D. Campbell: *Apolarity in the Galois fields of order* 2^n.

In this paper is derived the equation in hyperplane coordinates of the quadratic $\Sigma a_{ij}x_ix_j=0$, where $i, j=1, 2, \cdots, m$, where $j\geqq i$ and $a_{ji}=0$ if $j\neq 0$, also where a_{ij}, x_i, x_j all belong to the Galois fields of order 2^n. This equation in hyperplane coordinates differs according as m is even or odd. The discriminant Δ of $\Sigma a_{ij}x_ix_j=0$ is the determinant with rows $0, a_{12}, \cdots, a_{1m}$; $a_{12}, 0, \cdots, a_{2m}; \cdots; a_{1m}, a_{2m}, \cdots, 0$, if m is even, and one-half the determinant with rows $2a_{11}, a_{12}, \cdots, a_{1m}; a_{12}, 2a_{22}, \cdots, a_{2m}; \cdots; a_{1m}, a_{2m}, \cdots, 2a_{mm}$, if m is odd. Applying Δ to the pencil of quadratics $\lambda\Sigma b_{ij}x_ix_j+\mu\Sigma a_{ij}x_ix_j=0$) we get the condition for apolarity, which differs according as m is even or odd. The geometric interpretation of apolarity is different in the Galois fields of order 2^n from what it is in the ordinary real and complex domains or in the Galois fields of order p^n for $p>2$. (Received November 23, 1931.)

24. Professor A. A. Albert: *Normal division algebras of degree* 2^e *over an algebraic field.*

The author considers normal division algebras A of degree n over an algebraic field $R(x)$ for the case where n is a power of 2. The exponent of A is the least integer r such that A^r (the direct product power) is a total matric algebra. He proves that $r=2^e=n$ for this case. He also shows that the only self-reciprocal normal division algebras over an algebraic field are normal division algebras of order unity or four. As a consequence we have the very important theorem that the multiplication algebra of every pure Riemann matrix of the first kind is a field or a generalized quaternion algebra over a field. (Received November 17, 1931.)

25. Dr. Gordon Pall: *Partitionment of representations in indefinite binary quadratic forms.*

Representations of an integer in an indefinite binary quadratic form may be grouped into *sets* according to whether they are or are not transformable into one another by integral automorphs of the form. Assuming the least positive solution (t_1, u_1) of $t^2-du^2=4$, we obtain intervals containing one and only one representative of any given set of solutions of $ax^2+bxy+cy^2=n$. For example, in each set of solutions there is one and only one (x, y) satisfying $2|an|-2an \leqq dy^2 \leqq t_1|an|-2an$, $y \geqq 0$, except that there may be two solutions (x', y) and (x, y) if y is at the boundary of this interval. As the simplest application let us note that if $n>0$, the number of integral solutions (x, y) of $x^2-2y^2=n$, $|x| \geqq 2y \geqq 0$, is $g(n)=\Sigma(2|\nu)$, where ν runs over the positive divisors of n, except when n is a square or the double of a square, when this number is $g(n)+1$. (Received November 28, 1931.)

26. Dr. Gordon Pall: *A class of universal functions.*

The function $axy+bx+cy+d$, where a, b, c, d are integers, and $a \neq 0$, represents all integers for integral values of x and y, if and only if $b \equiv \pm 1$ or $c \equiv \pm 1 \pmod a$, or $a=6$, $b \equiv \pm 3$, $c \equiv \pm 2 \pmod 6$ or vice versa for b and c. A similar theorem holds for such functions which are positive for positive integers x and y and represent all positive integers for such values x and y. (Received November 28, 1931.)

27. Professor R. G. Archibald: *Properties of highly composite numbers.*

A highly composite number has been defined by S. Ramanujan as a positive integer the number of positive divisors of which exceeds that of every smaller positive integer. He has developed a theory of these numbers and obtained for them some properties analogous to the corresponding properties for primes; for example, if x is any quantity not less than unity, there is at least one highly composite number greater than x and not exceeding $2x$ (an analogue to Bertrand's postulate for primes). The present paper shows, on the other hand, that the analogy with primes may break down. It is proved, for example, that the sum of the reciprocals of all highly composite numbers is finite, and that every arithmetical progression $ax+b$, with a and b relatively prime, can contain only a finite number of highly composite numbers. More generally, it is shown that every arithmetical progression $ax+b$, with $a>b \geqq 1$, contains only a finite number of highly composite numbers. (Received November 28, 1931.)

28. Professor I. M. Sheffer: *A type of conformality in three dimensions.*

The class of conformal point transformations in three dimensions being rather restricted, as compared to the 2-dimensional case, one inquires if one can enlarge the class by demanding less. Thus, instead of asking for what transformations will angles between *all* pairs of curves transform conformally, let us put the following question: when will there exist a family of surfaces such that angles between all pairs of curves lying on any member of this family transform conformally? A transformation fulfilling such a condition is (we say) quasi-conformal; in this paper we examine the conditions for quasi-conformality. (Received November 24, 1931.)

29. Professor J. L. Synge: *The apsides of general dynamical systems.*

Hadamard (Journal de Mathématiques, (5), vol. 3 (1897)) discussed the relations of a dynamical trajectory to any system of surfaces $S=\text{const.}$ in the manifold of configurations. The method, which is based on the determination of the sign of d^2S/dt^2, is now more simply presented by the use of tensor calculus, and applied to obtain information concerning the distribution of apsides in the motion of systems of some important general types. The method is based on the use of the characteristic function F of the manifold (half the square of geodesic distance). The apsides considered are the points of contact of the trajectory with the equipotential surfaces and with a family of concentric spheres. (Received November 28, 1931.)

30. Dr. C. H. Dix: *A discontinuous dynamics.*

A theory of discontinuous motion on a lattice is developed in which the equations of motion are difference equations. These equations can be given in either the Lagrangian or the Hamiltonian form. In the Hamiltonian form they suggest a theory of canonical transformations and of Pfaffian difference forms which is developed by the author. (Received November 27, 1931.)

31. Dr. Eberhard Hopf (International Research Fellow): *On the time-average theorems in dynamics.*

J. v. Neumann has quite recently proved an important theorem on time-averages in dynamics using the spectral theory of unitary transformations in Hilbert space. Here a simpler and more elementary proof of v. Neumann's theorem is given. Let P denote points of an analytic manifold Ω, and let $P_t=T_t(P)$, $T_0(P)=P_0$, $T_sT_t=T_{s+t}$ be an analytic one-parameter group of (1, 1) transformations of Ω into itself. A Lebesgue measure is assumed to exist on Ω and be invariant under T_t. Then any quadratically integrable function $f(P)$ possesses a time-average, in the sense of convergence in the mean over Ω. The analyticity restrictions are unessential, and the time-average can be expressed as a space-average. (Received November 30, 1931.)

32. Dr. L. M. Blumenthal: *A complete characterization of proper pseudo d-cyclic sets of points.*

In characterizing the n-dimensional euclidean space R_n among general semi-metric spaces, Karl Menger introduced and characterized pseudo-euclidean sets of points (Mathematische Annalen, vol. 100 (1928), p. 120). These are sets of points such that each $n+2$ of them is congruent with $n+2$ points of the R_n, while the whole set is not congruent with a subset of the R_n. Since the R_n has the quasi-congruence order $n+2$, these sets consist of exactly $n+3$ points. In this paper we consider a d-circular space; that is, a space congruent with the convex circle of metrical diameter d. Such a space is *not euclidean*. A pseudo d-cyclic set of points is a set such that each three of the points is congruent to three points of the d-circular space, while the whole set is not congruent with a subset of this space. The set is proper if it does not contain a convex tripod. We futher assume that in case four points of a set are pseudo-linear, two of these points are diametral. The theorems contained in this paper completely characterize such sets. The circular space considered has *not* the quasi-congruence order three; and hence these sets do not consist of exactly four points. Pseudo d-cyclic quadruples are characterized, and it is then proved that *a proper pseudo d-cyclic set containing more than four points is equilateral.* (Received November 16, 1931.)

33. Dr. Rothwell Stephens: *Continuous transformations of abstract spaces.*

This paper is a study of continuous transformations on a general range, the topological space. In this space a continuous transformation is difficult to define, since all our intuitive notions cannot be satisfied. The Sierpinski definition seems the most satisfactory. Invariant properties for continuous, bicontinuous, and biunivocal transformations are determined. Finally the following problem is considered: Characterize the most general space such that there exists a non-constant continuous transformation to a given type of space. Necessary and sufficient conditions are found for the existence of continuous transformations of a space to neighborhood, accessible, and L spaces. (Received November 25, 1931.)

34. Mr. Alfred Korzybski: *A non-aristotelian system and its necessity for rigor in mathematics and physics.*

This paper gives an anthropological approach to the subject. Identity, a fundamental, false to fact, postulate of the aristotelian trilogy (the aristotelian, euclidean, newtonian systems), establishes elementalism; a non-aristotelian system rejects "identity," becomes non-elementalistic. The inter-relation between the non-aristotelian, non-euclidean, and non-newtonian systems (called non-systems) is given, also the essential differentiation between the un-speakable objective levels (ordinary objects, processes, action, functionings, immediate feelings, etc.) and verbal levels, establishing *structure*, defined in terms of relations or multi-dimensional order, as the only link between the two worlds. Structure becomes the only content of "knowledge." Semantic definition of number and mathematics in terms of relations explains why mathematics gives structure-knowledge. There is *one name* for happenings on the un-speakable levels and "mental pictures," a partial mechanism of identification. The two-valued aristotelian logic is a limiting case of a general many-valued logic of probability. Two-valued logic does not apply to processes, the objective world, the foundation of mathematics, the theory of infinity, etc. The many-valued logic covers all 1931 human needs, mathematics and mathematical physics included. (Received November 27, 1931.)

35. Professor J. L. Synge: *Thermostatic control.*

This paper studies the oscillations of the temperature of a room, the source of heat being automatically cut off when the temperature rises above an assigned level. The temperature of the room satisfies a linear differential equation of the second order with constant coefficients and with a discontinuous term. The problem has an exact analogue in the motion of a damped harmonic oscillator of the dead-beat type, subject to a constant disturbing force for a certain range of positions of the particle. It is found that the amplitude and the periodic time of the oscillations both tend to zero. (Received November 28, 1931.)

36. Dr. C. H. Dix: *Generalizations of time in dynamics.*

Possible generalizations of "time in dynamics" are considered. Instead of the Hamiltonian action integral we have a new integral in which the time is replaced by a monotone function $\tau(t)$ of a continuous parameter, t. The velocity is then the Stieltjes derivative of the coordinate $q(t)$ with respect to the $\tau(t)$. In the case of no force, the velocity is constant except for a set, T, of the t's for which $\int_T d\tau(t) = 0$. The case where the potential is not zero has also been considered. (Received November 27, 1931.)

37. Dr. H. B. Curry (National Research Fellow): *Some additions to the theory of combinators.*

The present paper contains two amplifications of the author's dissertation *Grundlagen der kombinatorischen Logik* (American Journal of Mathematics, vol. 52 (1930), pp. 509–536, 789–834). These are (1) a revision of the definition of normal form so that to every normal sequence there will correspond a unique normal combinator, (2) a generalization of the principal theorems on equality of combinators, so that the restriction that the combinators be "eigentlich" is removed. (Received November 25, 1931.)

38. Professor R. G. Putnam: *On sets of constant dimension.*

In this paper several theorems concerning constant dimensional sets (Menger, *Dimensionstheorie*, pp. 308–309) of separable spaces are proved. In particular, it is shown, for certain subsets of a compact separable space, that a necessary and sufficient condition for constancy of dimension is that each set be contained in a subset of the same dimension, which is both an F_σ and a G_δ. (Received November 23, 1931.)

39. Dr. Hassler Whitney (National Research Fellow): *Conditions that a graph have a dual.*

It is shown by purely combinatorial means that a graph has a dual if and only if it contains neither of two certain graphs as subgraphs. This, with a theorem of the author (Proceedings of the National Academy, vol. 17 (1931), p. 127, Theorem 15), gives a proof involving little of a point-set nature, of the case for finite graphs of a theorem of Kuratowski (Fundamenta Mathematicae, vol. 15 (1930), pp. 271–283). (Received November 28, 1931.)

40. Dr. Hassler Whitney (National Research Fellow): *Basic graphs.*

R. M. Foster has classified graphs of nullity $\leqq 6$, using the concept of basic graphs and a type of equivalence we shall call semi-homeomorphism. G is semi-homeomorphic with G' if it can be carried into G' by operations of subdividing and recombining arcs, altering the arrangement of the components of the graph, and breaking part of the graph away at two vertices, turning it around, and replacing it. A basic graph is a non-separable cubic graph not semi-homeomorphic with any graph with fewer arcs. All non-separable graphs can be obtained from basic graphs by dropping out arcs, at the same time letting their end vertices coalesce, and employing the above operations. A basic graph can be obtained from a basic graph of nullity one less by joining the mid points of two arcs by an arc. These theorems and others are proved by the author. (Received November 28, 1931.)

41. Dr. Hassler Whitney (National Research Fellow): *A set of topological invariants for graphs.*

The numbers in question are $p_i = \sum_j (-1)^{i+j} m_{R-j,N-i}$ (see a note by the author in the Proceedings of the National Academy, vol. 17 (1931), pp. 122–125; Theorem 8 should state $m'_{ij} = m_{R-j,N-i}$). On account of the equation just noted, we have, if G and G' are duals, $p_i = m'_i$, $m_i = p'_i$. An interpretation of the p_i is found analogous to that of the m_i in terms of broken circuits. (Received November 28, 1931.)

42. Dr. Hassler Whitney (National Research Fellow): *Characteristic functions and the algebra of logic.*

If A' is any subset of a set R', the characteristic function A of A' is defined to be 1 in A', and 0 in $R'-A'$ (de la Vallée Poussin, *Intégrales de Lebesgue*, p. 7). The characteristic function of $R'-A'$ is $1-A$, that of the common part of A' and B' is AB, and that of the sum of A' and B' is $A \dot{+} B = A+B-AB$. By means of these formulas, the ordinary relations in the logic of classes are easily worked out. Three normal forms for characteristic functions are studied. The notation and normal forms are important in other work of the author. (Received November 28, 1931.)

43. Professor F. R. Bamforth: *Divergent series and singular points of ordinary differential equations.*

In 1929 Professor Birkhoff and the author published under the above title a paper in which the setting of the problem in question was given and the statement was made that difficulties seemed to arise when any attempt was made to apply their method of solution to systems of differential equations of order greater than the third. In the present paper another method of solving this problem is given, such that it can readily be applied to systems of differential equations of any finite order. The problem for the third order is, by means of a transformation of class C^∞ on the dependent variables only, to reduce to the system of differential equations $dz_i/dt = m_i z_i$, $i=1, 2, 3$, every system of differential equations of the type $dx_i/dt = m_i x_i + P_i(x_1, x_2, x_3)$, $i=1, 2, 3$, in which (i) the P_i are power series in their arguments which have no terms of degree less than two and converge for $|x_1|$, $|x_2|$, $|x_3| \leqq r$, $r>0$, (ii) the m_i are real constants, $m_1>0$, $m_2>0$, $m_3<0$, which satisfy no equation of the form $m_i = p_1 m_1 + p_2 m_2 + p_3 m_3$ with any set whatsoever of positive integers p_1, p_2, p_3. (Received November 25, 1931.)

44. Professor W. H. Wilson: *A functional equation involving well known special cases.*

The addition theorems of sine and cosine, and the first two Cauchy equations are special cases of (1) $f(x+y)=af(x)f(y)+bf(x)g(y)+cf(y)g(q)+dg(x)g(y)$, where a, b, c, d are constants, and x and y are independent variables. This paper is designed to show how the functions $f(x)$ and $g(x)$ that satisfy equation (1) are related to functions that satisfy the four special cases named above. It is assumed that $f(x)\not\equiv 0$, but no such restriction as boundedness, continuity, differentiability, or integrability is assumed. In certain cases $g(x)$ is a constant multiple of $f(x)$ and in these cases the functions $f(x)$ and $g(x)$ are constant multiples of the function that satisfies the exponential equation. In the remaining cases the functions $f(x)$ and $g(x)$ lead to simultaneous equations from which we find that $f(x)$ and $g(x)$ are expressible in terms of functions satisfying the four equations named at the beginning. (Received November 23, 1931.)

45. Professor H. T. Davis: *A singular case of the Laplace differential equation of infinite order.*

In this paper the author considers the inversion of the Laplace differential equation of infinite order, $\sum_{n=0}^{p} A_n(d/dx) \rightarrow u(x)=f(x)$, $p>0$, for the singular case where $A_p(0)=0$. It is shown that the difficulties of inversion, which have been formally removed by a transformation due to O. Perron (Mathematische Annalen, vol. 84 (1921), p. 41), reside essentially in the nature of the operators $z^{-\nu}$ and $z^{-\nu}\log^m z$, $z=d/dx$. Adjoin these branch-point operators to the operators of ordinary integration and differentiation, and it is possible to give a complete account of the inversion of the equation. Examples are cited from the theory of integral and difference equations to illustrate the application of these operators. (Received November 27, 1931.)

46. Professor J. M. Thomas: *Pfaffian systems of species one.*

If the minimum number of differentials in terms of which a Pfaffian system of r equations can be expressed is $r+s$, the integer s is the *species* of the system. Completely integrable systems are of species zero. The next systems in order of simplicity are those of species one. It is shown how to recognize such a system and how to reduce it to a form involving the minimum number of differentials. The latter operation only requires the solution of ordinary differential equations. When the class of the system plus the number of equations in the maximum system of species zero contained by it is $2r+1$, the system admits a canonical form. (Received November 27, 1931.)

47. Professor Morris Marden: *A generalization of Weierstrass's and Fekete's mean-value theorems.*

The two principal results of the present paper may be stated as follows. (1) If on a curve C: $z=k(t)$, $a\leqq t\leqq b$, $f(z)$ and $g(z)$ assume any values within a given convex region F and an angular domain G (with vertex at $w=0$ and magnitude $\gamma<\pi$) respectively, the point σ as defined by the equation $\int_a^b f(z)\cdot g(z)dt=\sigma\int_a^b g(z)dt$ lies in the star-shaped region consisting of all points at which F subtends an angle $\phi\geqq\pi-\gamma$. Weierstrass's theorem is the case $\gamma=0$. (2) If $P(z)$ is a polynomial of degree n, and k_j and α_j are points of a convex region K and an angular domain A (with vertex at $z=0$ and magnitude $\gamma<\pi$) respectively, $P(z)$ assumes the value σ, as defined by the equation $\sum_1^m \alpha_j P(k_j)=\sigma\sum_1^m \alpha_j$, at least once in the star-shaped region consisting of all points at which K subtends an angle of $\phi \geqq (\pi - \gamma)/n$. When $\gamma = 0$ and $m = 2$, the theorem coincides with Fekete's, and when $\gamma = 0$ (no restriction on m) it gives a somewhat better approximation to the position of σ than does Nagy's extension of Fekete's theorem. (Received November 27, 1931.)

48. Dr. I. J. Schoenberg: *Concerning a certain class of linear inequalities connected with the theory of completely monotonic functions.*

It is well known that the most general solution of a system of linear homogeneous inequalities may be represented as a linear homogeneous combination with non-negative coefficients of a certain set of so-called *fundamental solutions.* This fact suggests that some infinite set of linear inequalities in infinitely many variables may have its general solution representable as Stieltjes integrals of the type $x_n = \int_a^b \phi_n(t) d\chi(t)$, $n = 0, 1, 2, \cdots$, with $\phi_n(t)$ continuous and $\chi(t)$ non-decreasing. This is indeed the case for completely monotonic sequences according to a theorem of Hausdorff. In a previous note *On finite and infinite completely monotonic sequences* (to appear in this Bulletin) the author has derived from this point of view Hausdorff's theorem mentioned above. In the present paper the same point of view is applied to obtain Hausdorff's results (Mathematische Zeitschrift, vol. 9, pp. 280–299) concerning his more general *sequences of moments* (Momentfolgen) and his *functions of moments* (Momentfunktionen) which include the so called Bernstein-Widder theorem on completely monotonic functions. These results are obtained by the use of a particular case of a method which may have further applications. (Received November 30, 1931.)

49. Professor L. W. Cohen: *A type of non-absolutely convergent integral.*

Let $f(x)$ be measurable on M, and $f^+(x)$, $f^-(x)$ be the positive and negative parts of $f(x)$. Consider $f_y^+(x) = \min[f^+(x), y]$, $f_y^-(x) = \min[f^-(x), y]$, $y \geqq 0$. Then $\int_M f^+(x)dx = \lim_y \int_M f_y^+(x)dx \leqq +\infty$, $\int_M f^-(x)dx = \lim_y \int_M f_y^-(x)dx \leqq +\infty$. The Lebesgue theory is concerned with the case where the inequalities hold in both cases. It is analogous to the theory of unconditionally convergent series. If the equalities hold in both cases, we have a conditionally convergent integral in the sense that, for any number A, there are sequences y_n', y_n'' such that $\int_{M(y_n')(y_n'')} f(x)dx = \lim_n [\int_M f_{y_n'}^+(x)dx - \int_M f_{y_n''}^-(x)dx] = A$. As a normal form of this definition we put $\int_M f(x)dx = \lim_y \int_M f_y(x)dx$, provided the limit exists, where $f_y(x) = f_y^+(x) - f_y^-(x)$. It is easily shown that this definition includes that of Lebesgue, that $\int_{-1}^1 (1/x)dx = 0$, that $g(y) = \int_M f_y(x)dx$ is uniformly continuous on $0 \leqq y < +\infty$, and that the "Hölder means" $\lim_{y_n}(1/y_n)\int_{y_n} dy_{n-1} \cdots (1/y_1) \cdot \int_0^{y_1} dy \cdot \int_M f_y(x)dx$ are regular summation processes in a sense analogous to that for non-absolutely divergent series. The paper also includes theorems giving necessary and sufficient conditions for the interchange of limit and integral under the above definition. (Received November 28, 1931.)

50. Dr. Nat Edmonson: *Note on the stereographic definition of plurisegments on the hypersphere.*

In a previous paper (Annals of Mathematics, (2), vol. 31 (1930), p. 13) the author of this note generalized a paper by Professors Evans and Bray (American Journal of Mathematics, vol. 49, p. 153) from three to n dimensions. In this generalization use was made of configurations on the hypersphere which were known as polar caps. The number of such polar caps for a given hypersphere depended on the dimension number of the hypersphere. The purpose of this note is to give an orthogonal parametric representation of the hypersphere such that the number of polar caps per hypersphere is reduced to one for all dimensions. This is accomplished by extending ordinary stereographic projection to n dimensions. All proofs in this new system are essentially the same as in the previous system of representing the hypersphere. (Received November 23, 1931.)

51. Dr. R. P. Agnew (National Research Fellow): *On Riesz and Cesàro methods of summability.*

Let (1) $\alpha_n = n^{-r}\sum_{k=0}^{n-1}(n-k)^r u_k$, (2) $\beta(t) = t^{-r}\sum_{k \leqq t-1}(t-k)^r u_k$, (3) $\gamma(t) = t^{-r} \cdot \sum_{k<t}(t-k)^r u_k$, and let σ_n denote the nth element of the Cesàro transform of order r. Then $\sum u_n$ is summable (1), (2), (3), or C_r to L if $\lim_{n\to\infty}\alpha_n = L$, $\lim_{t\to\infty}\beta(t) = L$, $\lim_{t\to\infty}\gamma(t) = L$, or $\lim_{n\to\infty}\sigma_n = L$. In each case r may be any complex constant except that when C_r is involved it may not be a negative integer. We show that, when $\mathcal{R}(r) < 0$, the familiar Riesz definition (3) does not constitute a useful method of summability, and that (2) is equivalent to C_r when $-1 < \mathcal{R}(r) < 0$. The following result, which is of interest in itself, is used in our proof: If $\mathcal{R}(r) < 0$ and $\lim_{n\to\infty} n^{-r}u_n = 0$, then $\lim_{n\to\infty}(\alpha_n - \sigma_n) = 0$. Further relations between (1), (2), (3), C_r, and closely related methods of summability, are obtained. (Received November 28, 1931.)

52. Dr. Hassler Whitney (National Research Fellow): *Note on Perron's solution of the Dirichlet problem.*

A simple proof of the theorem that the lower bound of all upper functions is harmonic is given (see Remak, Journal für Mathematik, vol. 156 (1926), p. 227). (Received November 28, 1931.)

53. Dr. D. G. Bourgin: *The problem of the sheet.*

The term "sheet" characterizes the limiting case of the thin plate under displacements large compared with the thickness when the bending stresses may be neglected. It is first shown that Hencky's (Zeitschrift für Angewandte Mathematik, vol. 1, pp. 81, 423) numerical solution by means of difference equations is invalid because of incorrect boundary conditions. An approximation method is developed based on expressing the non-linear equations as non-homogeneous linear equations. A direct treatment is made possible by the use of the stresses as the primary dependent variables rather than displacements which are usually used in elasticity problems. The solution involves the introduction of a harmonic function determined by the value of a sufficiently well behaved second derivative on the boundary (except for terms in xy and linear terms). The key boundary condition introduced for the clamped rectangular sheet is $\partial T_2/\partial y = \partial S_1/\partial y$ for the boundary $x = 0$, $x = a$, and a similar condition for $y=0$, $y=b$ (Love's notation). The analysis is in terms of the Green's function and the associated orthogonal functions. (Received December 4, 1931.)

54. Professor E. V. Huntington: *A new set of independent postulates for the algebra of logic.*

An abstract of this paper will appear in the February issue of the Proceedings of the National Academy of Sciences.

55. Professor H. A. Simmons: *Generalizations of the author's earlier results relative to the Diophantine equation $\sum(1/(x_1x_2, \cdots, x_r))=b/a$.*

Using notation of our earlier paper (Abstract 37-5-190), we first consider the equation (1) $\Sigma_{n,r}(1/x)+\lambda_{r+1}\Sigma_{n,r+1}(1/x)+ ,\cdots, +\lambda_s\Sigma_{n,s}(1/x)=b/a$, $a\equiv(c+1)b-1$, where r, s, n are positive integers with $r<s\leqq n$; b, c are positive integers; and the $\lambda_t(t=r+1,\cdots, s)$ are integers $\geqq 0$. Application of Kellogg's process (American Mathematical Monthly, Aug.–Sept., 1921, p. 300) to (1) leads to a solution in positive integers of (1), namely $x=w$, where (2) $w_t=1$ $(t=1,\cdots, r-1)$, $w_r=c+1$, $w_{t+1}=a[\Sigma_{t,t-r+1}(w)+\lambda_{r+1}\Sigma_{t,t-r}(w)+,\cdots, +\lambda_s\Sigma_{t,t-s+1}(w)]+1(t=r,\cdots, n-2)$, $w_n=a[\Sigma_{n-1,n-r}(w)+\lambda_{r+1}\ \Sigma_{n-1,n-r-1}(w)+,\cdots, +\lambda_s\Sigma_{n-1,n-s}(w)]$. In the cases where (i) $\lambda_t=1(t=r+1,\ \cdots, s)$, (ii) λ_{r+1} is an arbitrary positive integer and $\lambda_t=0$ for $(r+1)<t\leqq s$, we obtain for the w of (2) two theorems on maximum numbers which are analogous to results that were stated in the abstract of the previous paper. We also obtain the Kellogg solution of the equation $\Sigma_{t=r}^{s}[\lambda_t\Sigma_{n,t}(1/x)]=1$, where λ_r is a positive integer and $\lambda_t(r<t\leqq s)$ is an integer $\geqq 0$; and for this equation we obtain results analogous to those described for (1). (Received December 28, 1931.)

56. Mr. E. R. Lorch: *Elementary transformations.*

An elementary transformation on two variables of order one is a transformation defined by two elementary functions each of order one at most. Necessary and sufficient conditions are given in this paper that the inverse of an elementary transformation on two variables of order one be elementary and of order one. (Received December 9, 1931.)

57. Professor R. L. Jeffery: *k-dimensional sets in n-dimensional space.*

A paper by A. S. Besicovitch (Mathematische Annalen, vol. 98 (1927), p. 422) deals with the geometrical properties of linearly measurable plane sets of points. These sets are measurable in the sense of Carathéodory, and this fact plays a fundamental rôle in the various discussions. The present paper shows that the results of Besicovitch hold for sets which are not necessarily measurable. It is also shown that k-dimensional sets in n-dimensional space have properties analogous to those of one-dimensional plane sets: e.g., the upper density is not greater than unity, while the lower bound of the upper density is different from zero, and is a function of k. (Received December 3, 1931.)