NOTE ON THE NUMBER OF LINEARLY INDEPENDENT DIRICHLET SERIES THAT SATISFY CERTAIN FUNCTIONAL EQUATIONS*

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In a paper published in the Transactions of this Society,† I have determined for each positive integer a the conditions on the coefficients of certain linear combinations of the Dirichlet series

$$\sum_{\nu=0}^{\infty} (a\nu+b)^{-s}, \qquad (b=1, \cdots, a-1),$$

in order that the function $f(s)$ thus determined shall satisfy one or another of the four functional equations

$$\text{(I) and (II):} \quad f(s) = \pm (a)^{1/2}\left(\frac{2\pi}{a}\right)^{s}\sin\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi}f(1-s),$$

$$\text{(III) and (IV):} \quad f(s) = \pm (a)^{1/2}\left(\frac{2\pi}{a}\right)^{s}\cos\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi}f(1-s).$$

Three cases had to be considered; $1: a\equiv 2 \pmod 4$, which was completely solved; $2: a=4q$; $3: a=2m+1$. These two cases were carried so far as to determine the roots of the characteristic equations $D(k)=0$ and the probable multiplicities of these roots. As the multiplicity of each root determines the number of linearly independent functions $f(s)$ that satisfy a given one of the equations (I), $\cdots$, (IV), it is evidently important to give a precise solution of the problem.

This seems all the more desirable, since the problem was first solved for the case a a prime number, by E. Cahen in 1894,‡ and needs only to be completed in the one respect indicated to have a complete and rigorous solution for every positive integer value of a.

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† *Properties of functions represented by the Dirichlet series* $\sum(a\nu+b)^{-s}$, *or by linear combinations of such series*, vol. 31 (1929), pp. 322–344. I shall refer to this paper briefly by T.

‡ Annales de l'École Normale, (3), vol. 11; in particular, pp. 137–154.

The solution is very simple and consists in observing that the sum of the roots of $D(k)=0$ [T, pp. 333–339] is, in every case, $\sum c_\nu^2$ for the (A) equations [T, p. 332] and $\sum s_\nu^2$ for the (B) equations [T, p. 336],

$$c_\mu = \cos \frac{2\pi\mu}{a}, \qquad s_\mu = \sin \frac{(2\pi\mu}{a} .$$

These may be evaluated by Gauss' sums.

To illustrate: In case 2, $a=4q$, we have*

$$\sum_{\nu=1}^{2q-1} c_\nu^2 = \frac{\sqrt{a}}{2} - 1.$$

The roots of $D(k)$ are [T, p. 338] 0, -1, both simple, and $\sqrt{a}/2$, $-\sqrt{a}/2$ of multiplicities which we will denote by p and n respectively. Then,

$$\text{sum of roots of } D(k) = 0-1+\tfrac{1}{2}(a)^{1/2}(p-n)$$
$$= \frac{\sqrt{a}}{2} - 1 .$$

Moreover, since the degree of $D(k)$ is $2q-1$, we have

$$p + n + 2 = 2q - 1.$$

The solution of these two equations gives

$$p = q - 1, \quad n = q - 2,$$

which agree with the conjectured values [T, p. 328].

All the other cases may be treated in a similar manner and verify the results previously indicated without proof.

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* See, for example, E. Landau, *Vorlesungen über Zahlentheorie*, vol. I, p. 153.