5. *Conclusion.* It is worthy of remark that by means of the property (III), alone, the field of functions to which $F[f]$ is applicable may be considerably extended. *If* (III) *is valid,* $F[f]$ *may be extended to all functions f in R summable Lebesgue.* And if $\Phi_1(x) \equiv -\infty$, $\Phi_2(x) \equiv +\infty$, the functional may be extended to all functions summable Lebesgue.

In fact, under (III), the functional has the same fundamental property as integration with respect to sequences $f_n(x)$ with limit $f(x)$; if the absolute continuity of $\int f_n(x)dx$ is uniform in n and $\lim f_n(x) = f(x)$, it follows that $\lim F[f_n]$ exists and is independent of the choice of the sequence f_n. Hence $F[f]$ may be defined as the limit of $F[f_n]$. And thus in successive steps, and by writing $F[f_1] = F[f_2]$, if f_1 and f_2 differ only on a set of zero measure, the extension is completed.

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THE EVALUATION OF CERTAIN DEFINITE
INTEGRALS BY THE USE OF
PROBABILITY FUNCTIONS*

BY W. D. BATEN

1. *Introduction.* The object of this paper is to present three methods of evaluating certain definite integrals by using probability functions. The first method consists in finding the probability law or function for the sum of n independent variables, which are each subject to given probability laws, by Mayr's method† and comparing this with the probability for the same sum obtained by Dodd's method.‡ On equating the two results thus obtained one often finds the value of certain integrals.

Method II consists in finding the probability law for n independent variables, which is expressed as an integral, and then allowing n to be equal to 1.

Method III is founded on setting up a probability function containing the integral to be evaluated as a factor, determining a second probability function defined over the same interval, and then finding a third probability function for a chosen relation of the variables under consideration. The integral need not be a factor of the first function, but should arise somewhere in the process.

The first two methods, as far as the writer knows, seem to be new.

2. *Method I.* Consider the integral

$$(1) \qquad \int_0^\infty \frac{\cos{(ut - nk)}dt}{(q^2 + t^2)^{nr/2}},$$

where $k = r \tan^{-1}{(t/q)}$, and $r > -1$. Let x_i be taken arbitrarily out of the interval $(0, \infty)$ and let

$$\frac{q^r}{\Gamma(r)} x_i^{r-1} e^{-qx_i} dx_i$$

be the probability that x_i lies in the interval (x_i, x_i+dx_i). The probability function for the sum $x_1+x_2+x_3+ \cdots +x_n = u$ can be written in two different forms following Mayr's or Dodd's method. These will be denoted hereafter by $P_n(u)$ and $F_n(u)$, respectively. We may then write, omitting du,

$$P_n(u) = \frac{q^{nr}u^{nr-1}e^{-qu}}{(nr - 1)!},$$

$$F_n(u) = \frac{q^{nr}}{\pi} \int_0^\infty \frac{\cos{(ut - nk)}dt}{(q^2 + t^2)^{nr/2}}.$$

Equating these, we find

$$\int_0^\infty \frac{\cos{(ut - nk)}dt}{(q^2 + t^2)^{nr/2}} = \frac{\pi u^{nr-1}e^{-qu}}{(nr - 1)!}.$$

Evaluating the integral, we obtain

$$(2) \qquad \int_{-\infty}^\infty e^{-h^2 x(x-u)} dx.$$

Let x_i be taken arbitrarily out of the interval $(-\infty, \infty)$ and let the probability function for x_i be

$$\frac{h}{\sqrt{\pi}}e^{-h^2x_i^2}dx_i.$$

The probability function for the sum $x_1+x_2=u$ is

$$P_2(u) = \frac{h^2}{\pi}\int_{-\infty}^{\infty}e^{-h^2u^2}e^{-2h^2x(x-u)}dx, \quad \text{(Mayr's method)},$$

$$F_2(u) = \frac{h}{(2\pi)^{1/2}}e^{-h^2u^2/2}, \qquad \text{(Dodd's method)}.$$

Hence we may write

$$\int_{-\infty}^{\infty}e^{-2h^2x(x-u)}dx = \frac{\sqrt{\pi}}{h\sqrt{2}}e^{h^2u^2/2}.$$

Let x_i be taken arbitrarily out of the interval $(-\infty, \infty)$ and let the probability function for x_i be

$$\frac{2h}{\pi}\cdot\frac{1}{e^{hx_i} + e^{-hx_i}}.$$

The probability function for the sum $x_1+x_2=u$ is

$$P_2(u) = \frac{4h^2}{\pi^2}\int_{-\infty}^{\infty}\frac{e^{hu}e^{hx}dx}{(e^{2hu} + e^{2hx})(e^{2hx} + 1)},$$

$$F_2(u) = \frac{4h^2}{\pi}\int_{0}^{\infty}\frac{\cos ut\, dt}{(e^{\pi t/(2h)}+e^{-\pi t/(2h)})^2} = \frac{4h^2u}{\pi^2(e^{hu} - e^{-hu})};$$

hence

$$(3) \qquad \int_{-\infty}^{\infty}\frac{e^{hx}dx}{(e^{2hu} + e^{2hx})(e^{2hx} + 1)} = \frac{u}{e^{2hu} - 1}.$$

Let the probability function for x_i be $1/k$ for $-k/2 \leqq x_i \leqq k/2$, and zero elsewhere. The probability function for the sum $x_1+x_2=u$, by Mayr's method, is

$$P_2(u) = \begin{cases} 0, & \text{for } (-\infty < u \leqq -k), \\[2mm] \dfrac{u+k}{k^2}, & \text{for } (-k \leqq u \leqq 0), \\[2mm] \dfrac{-u+k}{k^2}, & \text{for } (0 \leqq u \leqq k), \\[2mm] 0, & \text{for } (k \leqq u < \infty); \end{cases}$$

and by Dodd's method it is

$$F_2(u) = \frac{4}{k^2\pi} \int_0^\infty \frac{\sin^2 (kt/2) \cos (ut)dt}{t^2} .$$

Hence we have

(4) $\dfrac{4}{k^2\pi} \displaystyle\int_0^\infty \dfrac{\sin^2 (kt/2) \cos (ut)dt}{t^2}$

$$= \begin{cases} 0, & \text{for} \quad (-\infty < u \leqq -k), \\[2mm] \dfrac{u+k}{k^2}, & \text{for} \quad (-k \leqq u \leqq 0), \\[2mm] \dfrac{-u+k}{k^2}, & \text{for} \quad (0 \leqq u \leqq k), \\[2mm] 0, & \text{for} \quad (k \leqq u < \infty). \end{cases}$$

In a similar manner, by using the probability law stated above for x_i, and the probability law for the sum $x_1+x_2+x_3$, or $x_1+x_2+x_3+x_4$, we get

$$\frac{6}{k^3\pi} \int_0^\infty \frac{\sin^3 (kt/2) \cos (ut)dt}{t^3}$$

$$= \begin{cases} 0, & \text{for} \quad \left(-\infty < u \leqq -\dfrac{3k}{2}\right), \\[2mm] \dfrac{1}{2k^3}\left(u + \dfrac{3k}{2}\right)^2, & \text{for} \quad \left(-\dfrac{3k}{2} \leqq u \leqq -\dfrac{k}{2}\right), \\[2mm] \dfrac{1}{4k^3}(-4u^2 + 3k^2), & \text{for} \quad \left(-\dfrac{k}{2} \leqq u \leqq \dfrac{k}{2}\right), \\[2mm] \dfrac{1}{2k^3}\left(-u + \dfrac{3k}{2}\right)^2, & \text{for} \quad \left(\dfrac{k}{2} \leqq u \leqq \dfrac{3k}{2}\right), \\[2mm] 0, & \text{for} \quad \left(\dfrac{3k}{2} \leqq u < \infty\right); \end{cases}$$

$$\frac{2^4}{k^4\pi} \int_0^\infty \frac{\sin^4 (kt/2) \cos (ut)dt}{t^4}$$

$$= \begin{cases} 0, & \text{for} \quad (-\infty < u \leqq -2k), \\[2mm] \dfrac{1}{6k^4}(u + 2k)^3, & \text{for} \quad (-2k \leqq u \leqq -k), \\[2mm] \dfrac{1}{6k^4}(-3u^3 - 6ku^2 + 4k^3), & \text{for} \quad (-k \leqq u \leqq 0), \\[2mm] \dfrac{1}{6k^4}(3u^3 - 6ku^2 + 4k^3), & \text{for} \quad (0 \leqq u \leqq k), \\[2mm] \dfrac{1}{6k^4}(-u + 2k)^3, & \text{for} \quad (k \leqq u \leqq 2k), \\[2mm] 0, & \text{for} \quad (2k \leqq u < \infty). \end{cases}$$

By continuing this process, a probability function for the sum of n independent variables is obtained, which is continuous and has different analytic expressions in different intervals of its existence, yet which can be expressed as a single definite integral.

3. *Method* II. If $f_i(x_i) = 3(1 - x_i^2)/4$ is the probability law for x_i for $-1 \leqq x_i \leqq 1$, and zero elsewhere, then the probability law for the sum $x_1 + x_2 + x_3 + \cdots + x_n = u$, is

$$(5) \qquad F_n(u) = \frac{3^n}{\pi} \int_0^\infty \frac{(\sin t - t \cos t) \cos (ut) dt}{t^{3n}}.$$

For $n = 1$, this integral is evidently $f_i(u)$. Hence

$$(6) \int_0^\infty \frac{(\sin t - t \cos t) \cos (ut) dt}{t^3} = \frac{\pi}{4}(1 - u^2), \; (-1 \leqq u \leqq 1).$$

In particular, for $u = 0, 1$, we have

$$(7) \qquad \int_0^\infty \frac{(\sin t - t \cos t) dt}{t^3} = \frac{\pi}{4},$$

$$(8) \qquad \int_0^\infty \frac{(\sin t - t \cos t) \cos t \, dt}{t^3} = 0.$$

If the probability function for x_i is $1/(2k)$, $-k \leqq x_i \leqq k$, and zero elsewhere, where k is a certain constant, then the law for the sum of n independent variables is, according to Dodd's method,

$$(9) \qquad F_n(u) = \frac{1}{\pi} \int_0^\infty \frac{(\sin kt)^n \cos (ut) dt}{k^n t^n} .$$

If we set $n = 1$ and $u = 0$, we find the well known integrals

$$(10) \qquad \int_0^\infty \frac{\sin (kt) \cos (ut) dt}{t} = \frac{\pi}{2}, \quad (-k \leqq u \leqq k),$$

$$(11) \qquad \int_0^\infty \frac{\sin (kt) dt}{t} = \frac{\pi}{2}, \qquad (k > 0).$$

Let the probability law for x_i be $(\cos x_i)/2$ for $-\pi/2 \leqq x_i \leqq \pi/2$, and zero elsewhere; the law for the sum of n variables is

$$(12) \qquad F_n(u) = \frac{1}{\pi} \int_0^\infty \frac{\cos^n (t\pi/2) \cos (ut) dt}{(1 - t^2)^n} .$$

If we set $n = 1$, this becomes

$$(13) \quad \int_0^\infty \frac{\cos (t\pi/2) \cos (ut) dt}{(1 - t^2)} = \frac{\pi}{2} \cos u, \quad (-\pi/2 \leqq u \leqq \pi/2).$$

In particular, with $u = 0$,

$$(14) \qquad \int_0^\infty \frac{\cos (t\pi/2) dt}{(1 - t^2)} = \frac{\pi}{2} .$$

If the probability law for x_i is

$$\frac{h}{\sqrt{\pi}} e^{-h^2 x_i} ,$$

for $(-\infty < x_i < \infty)$, then the law for the sum of the squares, $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = u$, is

$$(15) \qquad F_n(u) = \frac{h^n}{\pi} \int_0^\infty \frac{\cos (ut - nm) dt}{(h^4 + t^2)^{1/4}}, \qquad \tan 2m = \frac{t}{h^2} .$$

For $n = 1$, we find

$$(16) \qquad \frac{h}{\pi} \int_0^\infty \frac{\cos (ut - m) dt}{(h^4 + t^2)^{1/4}} = \frac{h e^{-h^2 u}}{(\pi u)^{1/2}} .$$

When written in full, this becomes

$$\frac{h}{2\pi} \int_0^\infty \frac{[(h^4+t^2)^{1/2}+h^2]^{1/2}\cos{(ut)} - [(h^4+t^2)^{1/2}-h^2]^{1/2}\sin{(ut)}\,dt}{(h^4+t^2)^{1/4}}$$

$$= \frac{he^{-h^2u}}{(\pi u)^{1/2}} \,.$$

4. *Method* III.   We shall now show how to evaluate the integral

$$(17) \qquad J = \int_0^\infty \frac{e^{-px}\sin{(qx)}\,dx}{x} \,.$$

Let

$$f(x) = \frac{1}{J}\cdot\frac{e^{-px}\sin qx}{x}, \qquad g(y) = e^{-y},$$

respectively, be the probability functions for the variables x and y for $0 \le x$, $y < \infty$, and let $y = xs$; then the probability function for s is*

$$P(s) = \frac{1}{J}\int_0^\infty \frac{e^{-px}\sin{(qx)}e^{-sx}x\,dx}{x} = \frac{1}{J}\cdot\frac{q}{q^2+(p+s)^2} \,.$$

Since $P(s)$ is a probability function for the interval $(0, \infty)$, we have $\int_0^\infty P(s)\,ds = 1$.   Therefore

$$\frac{1}{J}\int_0^\infty \frac{q\,ds}{q^2+(p+s)^2} = \frac{1}{J}\left(\frac{\pi}{2} - \tan^{-1}\frac{p}{q}\right) = \frac{1}{J}\left(\tan^{-1}\frac{q}{p}\right).$$

Hence

$$J = \int_0^\infty \frac{e^{-px}\sin{(qx)}\,dx}{x} = \tan^{-1}\frac{q}{p} \,.$$

Consider the integral

$$(18) \qquad \int_0^\infty \frac{ds}{1+s^4} \,.$$

Let

$$f(x) = \frac{\sqrt{2}}{\pi}\cdot\frac{\sin x}{\sqrt{x}}, \qquad g(y) = \frac{2}{\sqrt{\pi}}e^{-y^2},$$

where x and y lie in the interval $(0, \infty)$, and let $y = s\sqrt{x}$.   Then we have

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* See the second footnote on p. 433.

$$(19) \qquad P(s) = \frac{2\sqrt{2}}{\pi} \int_0^\infty \frac{e^{-xs^2}\sqrt{x}\,\sin x\,dx}{\sqrt{x}} = \frac{2\sqrt{2}}{\pi(1+s^4)} \cdot$$

Since we know that $\int_0^\infty P(s)ds = 1$, we may write

$$\int_0^\infty \frac{ds}{1+s^4} = \frac{\pi}{2\sqrt{2}} \cdot$$

We shall now prove that

$$(20) \qquad \int_0^\infty \frac{\cos x\,dx}{x+a} = \int_0^\infty \frac{xe^{-ax}dx}{1+x^2} \cdot$$

Let

$$f(x) = \frac{1}{J} \cdot \frac{\cos x}{x+a}, \qquad g(y) = e^{\pm y},$$

for $0 \leqq x,\ y < \infty$, where J is the integral on the left of the above equality. The probability law for s, where $y = (x+a)s$, is

$$(21) \quad P(s) = \frac{1}{J} \int_0^\infty \frac{\cos x\,e^{-(x+a)s}(x+a)dx}{(x+a)} = \frac{1}{J} \cdot \frac{se^{-as}}{1+s^2} \cdot$$

But $\int_0^\infty P(s)ds = 1$. Hence

$$\frac{1}{J} \int_0^\infty \frac{se^{-as}ds}{1+s^2} = 1,$$

or

$$(22) \qquad \int_0^\infty \frac{se^{-as}ds}{1+s^2} = J = \int_0^\infty \frac{\cos s\,ds}{s+a},$$

which agrees essentially with (20).

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