ON THE NATURE OF θ IN THE MEAN-VALUE THEOREM OF THE DIFFERENTIAL CALCULUS*

BY GANESH PRASAD

1. *Introduction.* If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b), the ends being included, than the relation

$$\text{(M)} \qquad f(x+h) = f(x) + hf'(x+\theta h), \qquad 0 < \theta < 1,$$

holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b); provided that *either* $f'(x)$ exists at every point inside the interval (a, b) *or* a certain less restrictive condition† is satisfied. In recent years the nature of θ has been studied by a number of writers‡ who start with the assumption that $f''(x)$ exists everywhere in the interval (a, b). The two theorems, which it is the object of this paper to formulate and prove, are believed to be new and hold even if $f''(x)$ does not exist everywhere. For the sake of clarity and fixity of ideas, I consider θ only as a function of h, assuming x to be a constant, say 0, in the theorem (M).

2. THEOREM I. *If $\theta(h)$ is single-valued and continuous, it is not necessarily differentiable for every value of h.*

PROOF. Take $f(x)$ to be the indefinite integral of a monotone, increasing and continuous function which has a differential coefficient everywhere in the interval (a, b), excepting the points

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† The condition of W. H. Young and G. C. Young, Quarterly Journal of Mathematics, vol. 40 (1909), p. 1; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 3d edition, 1927, p. 384; or the still less restrictive condition of A. N. Singh, Bulletin of the Calcutta Mathematical Society, vol. 19 (1928), p. 43.

‡ R. Rothe (Mathematische Zeitschrift, vol. 9 (1921), p. 300; Tôhoku Mathematical Journal, vol. 29 (1928), p. 145); T. Hayashi (Science Reports of the Tôhoku Imperial University, (1), vol. 13 (1925), p. 385); O. Szász (Mathematische Zeitschrift, vol. 25 (1926), p. 116).

of an everywhere dense set. Such a function is that given by T. Broden.* Denoting Broden's function by w, let

$$f(x) = \int_0^x w(t)dt,$$

and let the everywhere dense set be denoted by S; also let ξ stand for $h\theta$. Then it is easily seen that ξ is a single-valued and continuous function of h, and that, corresponding to each value of ξ, there is a value of h and only one value. Now (M) gives

$$f(h) = hf'(\xi) = hw(\xi),$$

whatever h may be.

Therefore, as $f'(h)$ exists,

$$\frac{d}{dh}\{hw(\xi)\}, \text{ that is, } w(\xi) + h\frac{dw}{dh},$$

must exist for every value of h. Thus, at any point $h=h'$ which corresponds to a point $\xi=\xi'$ of S, $d\xi/dh$ and, consequently, $d\theta/dh$ must be non-existent; otherwise $w'(\xi')$ will exist which is impossible.

Therefore it is proved that, for every value of h corresponding to which ξ is a point of S, $d\theta/dh$ is non-existent.

3. THEOREM II. *If $\theta(h)$ is single-valued, it is necessarily continuous for every value of h.*

PROOF. Assume, if possible, that $\bar{h}$ is a point of discontinuity of $\theta(h)$. Then, denoting the corresponding values of ξ and θ by $\bar{\xi}$ and $\bar{\theta}$ respectively, we have by (M)

$$f(\bar{h}) = \bar{h}f'(\bar{\xi}).$$

Now two possibilities arise: the discontinuity may be of the first kind or of the second kind.

(a) If the discontinuity is of the first kind, then there must be a sequence $\{h_n\}$, tending to $\bar{h}$, for which the corresponding sequence $\{\xi_n\}$ does not tend to $\bar{\xi}$ but to $\bar{\xi}'$ different from $\bar{\xi}$. Thus

$$f'(\bar{\xi}) = f'(\bar{\xi}').$$

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* Journal für Mathematik, vol. 118, p. 27; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 389.

So, for the same value of h, namely, $\bar{h}$, there are two values of θ, namely, $\bar{\theta}$ and $\bar{\theta}'$, which is absurd, since θ is single-valued.

(b) If the discontinuity is of the second kind, then there must be a sequence $\{h_n\}$, tending to $\bar{h}$, for which the corresponding sequence $\{\xi_n\}$ does not tend to any limit. Therefore two values k_1 and k_2 of h can always be found as near as we please to $\bar{h}$ such that the corresponding values η_1 and η_2 of ξ differ from each other by a quantity greater than a suitably prescribed positive quantity δ. But, from (M), $f(h)/h$ and, consequently, $f'(\xi)$ are continuous functions of h at $\bar{h}$. Therefore ξ must be multiple-valued at $\bar{h}$, which is absurd, since θ is single-valued.

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A NUMERICAL FUNCTION APPLIED TO CYCLOTOMY

BY EMMA T. LEHMER

A function $\phi_2(n)$ giving the number of pairs of consecutive integers each less than n and prime to n, was considered first by Schemmel.* In applying this function to the enumeration of magic squares, D. N. Lehmer† has shown that if one replaces consecutive pairs by pairs of integers having a fixed difference λ prime to $n=\prod_{i=1}^{t} p_i^{\alpha_i}$, then the number of such pairs (mod n) whose elements are both prime to n is also given by

$$\phi_2(n)=\prod_{i=1}^{t} p_i^{\alpha_i-1}(p_i - 2) .$$

As is the case for Euler's totient function $\phi(n)$, the function $\phi_2(n)$ obviously enjoys the multiplicative property $\phi_2(m)\phi_2(n) =\phi_2(mn)$, $(m, n)=1$, $\phi_2(1)=1$. *In what follows we call an integer simple if it contains no square factor* >1. For a simple number n we have the following analog of Gauss' theorem:

$$\sum_{\delta|n}\phi_2(\delta) = \phi(n), \tag{1}$$

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* Journal für Mathematatik, vol. 70 (1869), pp. 191–2.

† Transactions of this Society, vol. 31 (1929), pp. 538–9.