ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.†

67. Professor F. C. Jonah: *The Green's matrix and expansion problem for systems of integro-differential equations.*

This paper treats a system of integro-differential equations $Y'(x)-\{A(x)\lambda+B(x)\}\ Y(x)=F(x)+\phi(x)\Lambda(Y)+\int_a^b R(x,\ s)Y(s)ds$ with the boundary condition $L(Y)\equiv W^{(a)}Y(a)+W^{(b)}Y(b)+\int_a^b H(x)Y(x)dx=0$, where $F(x)$ is an n-dimensional vector, $\Lambda(Y)$ is a linear operator on the vector Y, linearly independent of $L(Y)$, and the remaining letters denote n-dimensional square matrices whose elements possess certain properties of continuity and integrability. Four distinct cases are treated. In each case the author has found a Green's matrix and has carried through the expansion problem, various cases being essentially different one from another. An auxiliary theory of vector integral equations is developed, and the notion of a pseudo-resolvent matrix is introduced.

68. Professor I. M. Sheffer: *Polynomial solutions of linear differential equations; expansions.*

The writer has considered, in previous communications to the Society, a theory of *sets* of polynomials. On the analytic side such sets generate linear differential equations of infinite order of a certain type. In the present paper we consider a class of such equations, namely those of finite order. We consider properties of the polynomial solutions as well as properties of the associated set of functions defined by the dual differential equation. The paper is also concerned with the problem of expanding analytic functions in terms of the polynomial solutions. It may be noted that when the order of the equation is two, some familiar polynomial systems are included, among them the Legendre polynomials.

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* In the future, abstracts of papers will appear under the heading given above, separately from the reports of meetings, as explained in the announcement of pages 1–2 of the January issue. Eventually, many abstracts will be printed in advance of the meeting at which the corresponding paper is presented.

† Thus, if an abstract numbered 238 is printed in issue No. 5 of volume 36, the cross-reference will be (Abstract No. 36–5–238).

69. Dr. C. O. Oakley: *Differential equations containing absolute values of derivatives.*

The equation $u''+p_1u'+p_2u+q_1|u'|+q_2|u|=\phi$ is studied, and the existence of solutions continuous in their first two derivatives is established by means of the system of associated linear equations. Oscillation, separation and comparison theorems are developed for the homogeneous equation. It is found that the zeros of two solutions may (depending upon the coefficients and upon the initial conditions) (1) separate singly or coincide as in the theory of linear equations; (2) separate by pairs; (3), as an intermediate case, alternately separate and coincide. Further it is shown that solutions may jump from one kind of separation to another.

70. Professors Einar Hille and J. D. Tamarkin: *On irregular kernels.*

The authors study the integral equation $y(x)=f(x)+\lambda\int_a^b K(x, t)y(t)dt$, where $K(x, t)$ belongs to L. Assuming $\int_a^b |K(x, t)|dt \leqq Q$, and $|\lambda|<1/Q$ and that $f(x)$ is integrable, they show the existence of an integrable solution defined by the Neumann series. This is the only bounded solution if $f(x)$ is bounded, but other integrable or even quadratically integrable solutions may exist. A sufficient condition for uniqueness is determined. The authors have also studied the distribution of the characteristic values for a kernel of the type mentioned and for kernels belonging to L^p, $p<2$.

71. Mr. O. J. Farrell: *On the expansion of harmonic functions in terms of harmonic polynomials.*

Let B designate the interior of a simple closed finite Jordan curve C in the (x, y)-plane. Let $\zeta(z)$, $z=x+iy$, be a function which maps B conformally onto the interior of the unit circle. Then it is found that there exists a set of harmonic polynomials $\{p_n(x, y)\}$ *belonging to the region* B, together with a set of functions $\{q_n(x, y)\}$ continuous and biorthogonal to the set $\{p_n(x, y)\}$ on every curve C_ρ: $|\zeta(z)|=\rho$ $(0<\rho<1)$, such that if $F(x, y)$ be an arbitrary function defined and integrable (in the sense of Lebesgue) on an arbitrary curve C_ρ, the series $\sum_{n=1}^\infty a_np_n(x, y)$, $a_n=\int_{C_\rho}F(x, y)q_n(x, y)ds$, converges within C_ρ, converges uniformly in every closed region interior to C_ρ, and thus defines a function harmonic within C_ρ. The convergence of this series is also studied when further restrictions, such as continuity, or continuity with boundedness of variation, are put on $F(x, y)$. Series of the form $\sum_{n=1}^\infty c_np_n(x, y)$, where the c_n are arbitrary constants, are included in this study. In the special case where C is analytic most of these properties have already been established by Walsh (Proceedings of the National Academy, vol. 13 (1927), pp. 175–180).

72. Professor W. A. Hurwitz: *The oscillation of a sequence.*

For any given sequence (x) of real or complex numbers $x_1, x_2, x_3, \cdots$, let $\Omega(x)=\limsup |s_m-s_n|$ as $m\to\infty$, $n\to\infty$. The quantity $\Omega(x)$, called the *oscillation* of the sequence, may be considered as measuring the deviation of the sequence from convergence. The following theorem is proved: In order that a regular transformation of the form $\sum_{k=1}^{n} a_{n,k}x_k$ may be such that we have $\Omega(y) \leqq \Omega(x)$ for every sequence (x) it is necessary and sufficient that $\sum_{k=1}^{n} |a_{n,k}| \rightarrow 1$ as $n \rightarrow \infty$.

73. Professors D. C. Gillespie and W. A. Hurwitz: *On sequences of continuous functions having a continuous limit.*

The authors prove a theorem on summability of sequences of functions. A typical special case is the following: If in the interval $a \leqq x \leqq b$, $s_n(x)$ is continuous, $|s_n(x)| \leqq M$, $s_n(x) \rightarrow s(x)$, and $s(x)$ is continuous, then there exists a totally regular transformation which carries the sequence into a new sequence $\{\sigma_n(x)\}$ such that $\sigma_n(x) \rightarrow s(x)$ uniformly. Incidentally this result provides a necessary and sufficient condition that the limit of a bounded sequence of continuous functions be continuous.

74. Professor R. L. Jeffery: *The integrability of a sequence of functions.*

Let $f \equiv f_1, f_2, \cdots$ be a sequence of functions summable on the measurable set E, and convergent to the summable function f_0. Vitali has shown (Palermo Rendiconti, vol. 23, p. 137) that the equi-convergence of the sequence of integrals is necessary and sufficient for "complete integrability." This condition is sufficient but not necessary for "integrability," that is, that $\lim_{n\rightarrow\infty}\int_E f_n dx = \int_E f_0 dx$. In this paper we have obtained conditions which are both necessary and sufficient for integrability, the most general of which is as follows: Let $g \equiv g_1, g_2, \cdots$ be any sub-sequence of f. Let $\rho(g, n, \delta)$ be the least upper bound, $\sigma(g, n, \delta)$ the greatest lower bound of $\int_e g_i dx$, $(i=1, 2, \cdots, n)$, for all possible sets e with $me < \delta$. It is then necessary and sufficient for the integrability of f that corresponding to every g of f and every $\epsilon > 0$ there is a number $\delta_{g\epsilon} > 0$ such that for every $\delta < \delta_{g\epsilon}$ it is possible to find $n = n_\delta$ such that for $n \geqq n_\delta$ we have $|\rho(g, n, \delta) + \sigma(g, n, \delta)| < \epsilon$. If for every g both $\rho(g, n, \delta)$ and $\sigma(g, n, \delta)$ are bounded in n for some δ, necessary and sufficient conditions can be obtained in more concise form.

75. Dr. T. H. Gronwall: *Some series expansions in the lattice theory of crystals.*

The application of quantum considerations to the lattice theory of crystals leads to the summation of Eigenfunktionen of the hydrogen type with their origins at the different lattice points. The present paper gives a general method for reducing these sums to a form which is suitable for numerical calculation. An example of the type of formulas arrived at is the following, referring to the simple cubic lattice:

$$\sum_{l_1, l_2, l_3, =-\infty}^{+\infty}{}' e^{-\lambda(l_1^2+l_2^2+l_3^2)^{1/2}} = \frac{8\pi}{\lambda^3} - 1 + \frac{1}{2\pi^3}\sum{}' \frac{\lambda}{((\lambda^2)/(4\pi^2+l_1^2+l_2^2+l_3^2)^2},$$

where the summations run through all integers except the combination $l_1 = l_2 = l_3 = 0$.

76. Professor A. D. Campbell: *Rectangular axes associated with a curve in four-dimensional space.*

Consider a curve $x_1=f_1(s)$, $x_2=f_2(s)$, $x_3=f_3(s)$, $x_4=f_4(s)$, where s is the arc. The tangent at a point P has the direction cosines $\alpha_{1i}=x_i'$, where $i=1, 2, 3, 4$, and x_i' means dx_i/ds. One normal at P has the direction cosines $\alpha_{2i}=\rho_1 x_i''$ where $1/\rho_1^2=\sum x_i''^2$. The normal to the osculating hyperplane at P with direction cosines α_{3i} and a third normal at P that is perpendicular to these other two normals are now taken. Using these four mutually perpendicular lines, the author obtains in this paper direct generalizations to four-dimensional space of the Frenet-Serret formulas and of the moving trihedral for curves in three-dimensional space. In these generalizations a radius of curvature ρ_1 appears and also ρ_3, where $1/\rho_3^2=\sum\alpha_{3i}'^2$. An article by Thomas Craig, *Displacements depending on one, two, and three parameters in a space of four dimensions* (American Journal of Mathematics, vol. 20 (1898), pp. 135–156) treats the same subject; but in it no use is made of the osculating hyperplane, the normals, or the curvatures of the curve on which the origin of the moving axes is supposed to be situated.

77. Professor T. R. Hollcroft: *Equivalence in hyperspace.*

In this paper, formulas are derived for the equivalence of the following manifolds, each considered as occurring to any given multiplicity on i hypersurfaces of given order in i dimensions: (1) linear manifolds of any given dimension; (2) lines (planes) with a given number of points (points or lines) of intersection; (3) plane (space) curves of given order (order and rank); (4) surfaces of given order.

78. Professor C. A. Rupp: *Linear spaces associated with hyperquadrics.*

When the coefficients of a general linear homogenous equation in $n+1$ variables are functions of the coefficients of a general symmetric quadratic homogeneous equation in $n+1$ variables, there exist geometric relations between the hyperplane represented by the linear equation, the hyperquadric represented by the quadratic equation, and the coordinate simplex. These geometric relations are displayed for several particular cases.

79. Professor F. R. Sharpe: *Plane involutions of order three or four.*

Let $|C|\equiv a_1C_1+a_2C_2=0$ be a pencil of curves of grade n and genus p and $\bar{C}$ a curve of genus $\bar{p}$, determined by its basis points, which meets $|C|$ in $t-n$ variable points. If $t=p+1$, then $|C|+\bar{C}$ is a net of curves of genus $p+\bar{p}+t-n-1$ and superabundance $p+\bar{p}-n$ defining an involution of order t. The cases $t=3$ or 4 are considered in detail.

80. Professor F. R. Sharpe: *The mapping of monoidal involutions.*

Monoidal involutions with vertex O were reduced by Montesano to 5 types according as the lines through O are each invariant or interchanged in pairs by one of the four types of ternary involution. It is shown that the last four types can all be mapped on the same type of variety in four-way space.

81. Professor Virgil Snyder: *On an involutorial transformation found by Montesano.*

The involutorial transformation in question is of order 7, and belongs to a general linear line complex, each line of which contains three pairs of conjugate points. Various systems of invariant surfaces are derived, and a number of features established that suggest that it may be irrational.

82. Professor J. H. Neelley: *Notes on the rational plane oscnodal quartic curve.*

An article in the July–August, 1929, issue of this Bulletin proposed a theorem concerning the singularity in question which we have found to be false. In its stead this paper developes several new theorems which cover the whole field of the oscnode for the curve R_2^4. New necessary and sufficient conditions for the singularity are developed.

83. Dr. Oscar Zariski: *On the moduli of algebraic functions possessing a given monodromy group.*

This paper deals with an application of the Galois theory to algebraic geometry. All transitive groups of substitutions on n letters are divided, with respect to a given genus p, into two classes, *special* and *non-special* groups, a group G being in the first or the second class according as the n-valued algebraic functions of genus p possessing G as monodromy group are of special or general moduli. For instance, as was shown by the author elsewhere, all solvable groups are special for any $p>6$. Given a complete continuous system Σ of n-valued algebraic functions y having G as monodromy group, the author studies the problem of evaluating the number of birational moduli of the functions of Σ. On the algebraic curve ϕ, on which the generic function y of Σ corresponds to a linear involution g_n^1, *virtually complete linear series* $\|g_m\|$ are defined with *neutral sets of points* belonging to the jacobian set of the g_n^1. The notions of special virtual series and of index of specialty of a virtual series are also defined. In particular the consideration of the two-fold of the series g_n^1 leads to the following theorem, which generalizes a theorem stated by Severi for simple branch points: *The functions y of Σ depend on $3p-3-i$ moduli, where i is the index of specialty of the virtually complete series* $\|2g_n\|$.

84. Mr. A. P. Mellish: *An application of vector analysis to the theory of curves of constant width.*

In this paper the fundamental properties of curves of constant width are developed by means of vector analysis. This method may also be applied to surfaces of constant width.

85. Professor D. J. Struik: *Invariant treatment of ovals.*

Many theorems on ovals have been proved by methods such as the theory of Fourier series, that are not invariant under rotations and translations. An attempt is made to give a vector analytic treatment of this part of geometry.

86. Professor M. S. Knebelman: *Content-preserving transformations.*

This paper deals with extended point transformations which preserve the content of every k-cell in an n-dimensional Riemann space. It is shown that if there exists such a transformation for a given $k<n$, it must be a motion of the space into itself and therefore the content of every cell of every dimensionality is preserved. This theorem is not true if $k=n$, since in every n-dimensional Riemann space there exist transformations preserving the volume of every n-cell and not preserving the content of the cells of any lower dimensionality. These transformations are defined by a vector density whose divergence vanishes.

87. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau for any rectifiable contour in n-dimensional euclidean space.*

The author has introduced into the problem of Plateau (see abstracts in this Bulletin, vol. 33, pp. 143, 259; vol. 34, p. 405; vol. 35, p. 292, vol. 36, p. 49–50) the functional $A(g)=(1/(16\pi))\cdot\int_0^{2\pi}\int_0^{2\pi}\sum_i[g_i(\theta)-g_i(\phi)]^2 d\theta\, d\phi/\sin^2\frac{1}{2}(\theta-\phi)$ $=(\pi/2)\sum_{m=1}^{\infty}m(\sum_i a_{im}^2+\sum_i b_{im}^2)$, where the range of the argument g consists of all representations $x_i=g_i(\theta)$, $i=1, 2, \cdots, n$, of the given contour as topological image of the unit circle, and a_{im}, b_{im} denote the Fourier constants of g_i. Improper representations, in which entire arcs of the contour correspond to points of the circle or vice versa, must be included to produce a *compact* set $[g]$ and so permit the application of the results of Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906), pp. 1–74). $A(g)$ is either finite or $+\infty$; but if the contour is *rectifiable*, $A(g)$ surely has a finite value for every parameter θ such that the arc-length s obeys with respect to θ a Lipschitz condition of order $>\frac{1}{2}$. $A(g)$ is readily shown to be lower semi-continuous and hence *attains* its minimum value. Reasons are adduced for rejecting the possibility that the minimizing representation g^0 be improper. The required minimal surface is the harmonic surface determined by the boundary values $g_i^0(\theta)$ (Poisson's integral). This is the first solution of the problem of Plateau for a contour of the specified degree of generality.

88. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau when the contour is an arbitrary Jordan curve in n-dimensional euclidean space*; I.

From rectifiable contours we can pass to any Jordan contour by a variety of limit processes. For instance, Montel's principle might be employed. Better, suppose $x_i=f_i(t)$ to represent the given contour in some fixed parametrization; then we may express the continuous functions $f_i(t)$ as the limits of their Fejér trigonometric polynomials, and so obtain a sequence of rectifiable contours Γ_m: $x_i=S_{im}(t)$, $m=1, 2, 3, \cdots$, tending to the given contour Γ as limit. By the preceding paper, each Γ_m has a parameter θ_m giving the solution of the Plateau problem for Γ_m, θ_m being derivable from t by a topological transformation T_m of the unit circle into itself. Since the totality of autohomeomorphisms of the unit circle forms a compact set, there exists a subsequence of transformations T_m converging towards a limit T, provable to be a proper topological transformation of the unit circle into itself. Let θ denote the parameter on Γ derived from t by the transformation T, and let $x_i = g_i(\theta)$ be the equations of Γ in terms of this parameter; then the harmonic surface determined by $g_i(\theta)$ is the required minimal surface. The remark at the end of the preceding abstract applies here a fortiori.

89. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau when the contour is an arbitrary Jordan curve in n-dimensional euclidean space*; II.

An alternative method is to introduce the functional $A_\rho(g) = (\pi/2) \cdot \sum_{m=1}^{\infty} m(\sum_i a_{im}^2 + \sum_i b_{im}^2)\rho^{2m}$, involving the parameter ρ; $0 \leqq \rho < 1$. Under the specified restrictions on ρ, this functional is finite-valued for all g, the convergence of the infinite series being assured by Parseval's theorem. The functional is moreover *continuous*, and therefore will attain its minimum value if the range of g is any compact set. In order to avoid the degeneration of the minimizing g to the one for which every a_{im}, $b_{im} = 0$ ("extraordinary" representation, in which the whole unit circle corresponds to a point of the contour and vice versa), we require g to make three fixed points of the contour correspond to three fixed points of the circle; it is shown that eventually no generality is lost by this restriction. Let g_ρ minimize $A_\rho(g)$; since the set $[g]$ is compact, ρ can be made to tend to 1 so that g_ρ tends to a limit g_1, proved to be a proper representation and to give the required minimal surface. A similar method can be based on the functional $A_k(g) = (\pi/2)\sum_{m=1}^{k} m(\sum_i a_{im}^2 + \sum_i b_{im}^2)$, also expressible as a double integral. This finite, continuous functional attains its minimum for a certain g_k; and g_k tends to a limit furnishing the required solution if k tends suitably to ∞.

90. Dr. Jesse Douglas: *The problem of Plateau and the theorem of Osgood-Carathéodory on the conformal mapping of Jordan regions.*

The author was the first to point out (from the very beginning, April, 1926, of his work on the Plateau problem; see this Bulletin, vol. 33 (1927), pp. 143, 259) that the Plateau problem ought to be formulated so as to demand not merely the minimal surface but also a conformal representation of this surface on the interior of a circle; and that in this formulation *the Plateau problem includes the Riemann conformal mapping problem as the special case* $n = 2$, where n denotes the least number of dimensions of a euclidean space containing the given contour. Since in the preceding papers, the minimal surface together with its conformal representation is derived from a topological correspondence between the given contour and the unit circle, we have as an *immediate* corollary the following well known theorem of Osgood-Carathéodory (W. F. Osgood and E. H. Taylor, Transactions of this Society, vol. 14 (1913), pp. 277–298; Carathéodory, Mathematische Annalen, vol. 73 (1913), pp. 305–320): A conformal correspondence between the interiors of any two Jordan curves induces by continuity a one-one continuous correspondence between the Jordan curves.

91. Professor Virgil Snyder and Dr. Marguerite Lehr: *Generating involutions of infinite discontinuous Cremona groups of S_4 which leave V_3 invariant.*

This joint paper gives the equations and the characteristic properties of the transformations outlined in Professor Snyder's Presidential Address, Article 4, in this Bulletin, vol. 35, pp. 612–615.

92. Professor F. R. Sharpe: *Involutions of order n with an $(n-2)$-fold line and their mapping.*

The involutions mentioned in the title have been discussed by Montesano (Lincei Rendiconti, (4), vol. 5_2, pp. 123–130), who showed that the planes through the lines are interchanged in pairs and that between each pair there is a quadratic Cremona correspondence. It is here shown that this correspondence can be expressed by two bilinear equations which determine the analytical form of the involution and by their aid the involution can be mapped on a variety in five-way space.

93. Professor Harold Hotelling: *Spaces of statistical parameters.*

For a space of n dimensions representing the parameters $p_1, \cdots, p_n$ of a frequency distribution, a statistically significant metric is defined by means of the variances and covariances of efficient estimates of these parameters. Such a space, for the ordinary types of distributions, is always curved. For the two parameters of the normal law the manifold may be represented in part as a surface of revolution of negative curvature, with a sharp circular edge. On this surface, variation of the dispersion is represented by moving along a generator. For a Pearson Type III curve of any given shape the same surface occurs. For the unrestricted Type III curve there are three parameters; their space is investigated. Certain metrical properties which hold in general for spaces of statistical parameters are given.

94. Professor F. M. Weida: *The valuation of a continuous survivorship annuity with a continuous refund of an arbitrarily assigned part of the purchase price.*

The present value $_{-xy}$ of the continuous refund is the present value of a continuously decreasing insurance beginning with an amount $f\bar{s}_{xy}=\bar{s}'_{xy}$, $(0 \leqq f \leqq 1,\ \bar{s}_{xy}=\bar{a}_x+\bar{a}_y-\bar{a}_{xy}+\bar{r}_{xy})$, and decreasing continuously until there is insured an amount zero. This present value has been expressed as a definite integral whose lower limit is zero and whose upper limit is $\bar{s}'_{xy}$. When we are dealing with Makeham's first modification of the law of Gompertz as our law of mortality, the problem stated above admits a solution which involves the gamma function, the incomplete gamma function, the hypergeometric series, and certain exponential and logarithmic functions. In order to obtain a result that is more practicable for use and more adaptable to numerical computation, it has been found that the mortality table can be fitted sufficiently closely by using a best fitting parabola in the sense of the theory of least squares. The desired value of $\bar{s}_{ww}$ is obtained by a process of iteration from the following formula: $\exp(-\delta\bar{s}'_{ww}) = [(1/f)\bar{s}'_{ww} - 2\bar{a}_w + \bar{a}_{ww}] \cdot [A_1\bar{s}'^3_{ww} + A_2\bar{s}'^2_{ww} + A_3\bar{s}'_{ww} + A_4]^{-1}$. It has been found that for the purpose intended we may assume the DeMoivre hypothesis for an interval of t_1 years, where $t_1 \geqq s'_{ww}$, and we obtain a result readily adaptable to numerical computation.

95. Professor J. B. Reynolds: *Inextensible chains on fixed plane curves.*

In this paper equations are set up for the motion of an inextensible chain on a rough plane curve. By a proper choice of variables and constants, these equations are shown to apply to several cases in statics and kinetics usually treated separately.

96. Professor E. G. Keller: *Certain theorems of periodic orbits applied to synchronous machines.*

In this paper certain theorems pertaining to differential equations as employed in the study of periodic orbits are applied to the differential equations of current flow in synchronous machines. For example, at short circuit, the differential equations, expressed in normal form, are $[(c/2-a)+(c/2-b)\cos 2t](di/dt)-(r-2b\sin 2t+c\sin t\cos t)i-(\sin t+R\cos t)I=-RI_0[\cos t]$ $[(c/2-a)+(c/2-b)\cos 2t](dI/dt)+[2bc\sin 2t\cos t-rc\cos t-c(a+b\cos 2t)\sin t]i-[c\sin t\cos t+R(a+b\cos 2t)]I=-RI_0(a+b\cos 2t)$, where only i, I, and t are variables. A fundamental set of solutions i_1 and i_2 is obtained by reducing the equations to two simultaneous integral equations. The transformation $i=i_1I_1$, $I=i_2I_1+I_2$ leads to the second set of solutions of the left members. The periodic solutions are obtained by the method of variation of parameters.

97. Professor B. F. Dostal: *On some applications of number-theoretic functions to electrical engineering problems.*

In the general problem of superposing prescribed sets of current or voltage distributions upon a prescribed set of points of a conducting electric line or network, the distributions are determined by the hyperbolic functions of the complex hyperbolic angle subtended by the line, section, or element concerned. Under certain conditions these functions assume the values of a characteristic sequence or spectrum of special values; as, for example, real integers. A. E. Kennelly was the first to show the manner of their production in the natural or artificial electric line. The object of the present paper is to show that the sequences are representable and replaceable by Lucas functions, as the difference equations for both, $x^2=ax-b$ in symbolic form, are identical, and so also are their generating functions. The employment of Lucas functions in such superposition problems not only renders theoretical and numerical calculations more convenient, but opens the field for the introduction of other number-theoretic functions to aid in the solution of problems where non-integer distributions occur.

98. Mr. R. L. Peek, Jr.: *On the solution of certain cases of the general equation of diffusion.*

For diffusion in substances for which the conductivity and diffusivity are functions of temperature, the partial differential equation which determines the temperature is non-linear in character. For a certain class of boundary conditions the introduction of an auxiliary function of time and distance reduces this equation to an ordinary differential equation of the second order, in which the temperature and the auxiliary function appear as dependent and independent variables respectively. As this resulting equation can be solved by methods of approximation, the solution to the original equation can be thus obtained. The auxiliary function mentioned exists only for certain boundary conditions, and when it does exist, is not unique. It is shown that when this function exists the solution to the corresponding case in which the conductivity and diffusivity are constant (and the partial differential equation is therefore linear) is itself an auxiliary function of the type described. Thus, for the class of boundary conditions in question, the non-linear partial differential equation connecting temperature with time and distance can always be replaced by an ordinary differential equation relating the temperature to the temperature that would exist if the partial differential equation were linear. This method is shown to apply to the case of the semi-infinite solid.

99. Professor D. J. Struik: *The Dirac theory and sedenions.*

M. S. Vallarta and the author have indicated the connection of Dirac's theory of the spinning electron and the hypercomplex number system of the sedenions or quadriquaternions (Journal of the Franklin Institute, April, 1929). A paper by J. A. Schouten (Amsterdam Proceedings, 1929) showed this more in detail. In the present paper it is shown how Eddington's mathematical introduction into the Dirac theory (Proceedings of the Royal Society, 1928, 1929) is simplified through the use of sedenions, and their relation to quaternions and to the hypercomplex number system belonging to the "Vierergruppe" is indicated.

100. Professor Philip Franklin: *Dynamical systems with integrals quadratic in the velocities.*

Motions of a particle in a plane under conservative forces, possessing an integral quadratic in the velocities distinct from the energy integral, were discussed by Darboux. He showed that *in general* they were problems of Liouville's class, when suitable confocal ellipses and parabolas were taken as parametric curves. In this note, the special cases are investigated, and the complete result obtained that in all cases the problem is of Liouville's class when suitable parametric curves are introduced. These are either as found by Darboux, or are the limiting cases of confocal parabolas, polar coordinate lines, or cartesian coordinate lines. The relation of the result to a theorem of Birkhoff's on conditional integrals is indicated.

101. Professor Oystein Ore: *Some investigations on finite fields.*

This paper contains a series of theorems on finite fields, and applications to various problems concerning higher congruences.

102. Mr. B. F. Groat: *Newtonian similarity.*

Newton's inclusive Theorem of Similitude, *Principia* II, Proposition XXXII, Theorem XXVI, may be applied to construct working hydrodynamic models using "similarity" in its correct sense for space-, force-, and time-ratios.

103. Mr. W. R. Thompson: *The powers of a given ideal of a field which may divide exactly the discriminant of a relative field of mth degree.*

The present problem is an extension to relative fields of the problem whose solution was presented at the meeting of the Society in New York in October, 1929. If $k(\theta)$ is an algebraic field, p a rational prime greater than 1, P a prime ideal divisor of p of order e in $k(\theta)$, and D_k the discriminant of a relative field of mth degree, let $E \geqq 0$ be a rational integer such that D_k is exactly divisible by P^E. Furthermore, let $E_{p,e}^{(m)}$ be the class of possible values of E. Then Ore (Mathematische Annalen, vol. 97 (1927), p. 569) has given the maximum value of E, which we shall designate by $M_{(m,p,e)}$. The following is a solution of the problem of ascertaining what other values E may have for such a relative field. As will be apparent, this depends on m, p, and e only. For $p>2$ and α any positive rational integer, if $m=p^\alpha$ then $E_{p,e}^{(m)}=0, \cdots, M_{(m,p,e)}$, except all numbers of the form $\alpha e p^\alpha - 1 - g p^\alpha$, where g is a rational integer and $0 \leqq g < e$; if $m=p^\alpha+1$, then $E_{p,e}^{(m)}=$ the union of $(p-1)$ and $E_{p,e}^{(m-1)}$ (but $E_{p,e}^{(m-1)}$ includes $(p-1)$ unless $\alpha=1$ and $p>2$); in every other case $E_{p,e}^{(m)}=0, \cdots, M_{(m,p,e)}$. And for $p=2$ the result is formally the same as for $p>2$ except that the odd numbers less than $2e$ are never components of $E_{2,e}^{(m)}$.

104. Professor C. G. Latimer: *Certain identities in theta functions.*

In a former paper a class of identities in theta functions was obtained by employing sets of integral elements in generalized quaternion algebras. Jacobi's fundamental identity was included in this class, the algebra in this case being that of ordinary quaternions. In this paper, Schroeter's identity is obtained in a similar manner, and also several other identities which are believed to be new. Sets of algebraic numbers and of generalized quaternions, which are not in general integral, are employed.

105. Professor C. R. Adams: *On multiple factorial series.*

In this paper an investigation is made of the essential convergence properties of multiple factorial series. For simplicity of statement we confine ourselves to the double series

(1): $\sum_{m,n=0}^{\infty} a_{mn}, \qquad a_{mn}=m!n!c_{mn}/[x(x+1)\cdots(x+m)y(y+1)\cdots(y+n)]$,

in which the c_{mn} are constants and x and y are complex variables. Employing the Pringsheim definition of convergence, we prove theorems analogous to

Theorems I–V, VII of Landau's fundamental paper on the theory of simple factorial series (Münchener Sitzungsberichte, 1906, pp. 151–218). For this purpose it is necessary to introduce the assumption that $S_{ij}=\sum_{m=0}^{i}\sum_{n=0}^{j}a_{mn}$ is a bounded function of i and j for particular values of x and y for which (1) converges. It is shown, for example, that the region of convergence *with* S_{ij} *bounded* of (1) is a pair of related right half-planes. The proof of these theorems follows the general lines of Landau's work making essential use of results obtained by C. N. Moore in his recent important paper on convergence factors in multiple series (Transactions of this Society, 1927, pp. 227–238). Additional theorems are derived by generalization of a portion of Nörlund's work on simple factorial series (Acta Mathematica, 1914, pp. 327–387).

106. Professor Marston Morse: *A theory of periodic extremals.*

The problem is defined on an n-dimensional manifold. The integrand is positive, regular, and homogeneous, in the usual way. With each closed extremal the author associates an invariant integer, called *the type number*. This integer characterizes the essential relations of the closed extremal to neighboring closed curves. In the two-dimensional case it is related to the Poincaré rotation number. It is the number of conjugate points to a given point on one circuit of the extremal plus an integer called *the order of concavity* of the extremal. The theory of closed extremals in the large is developed with the aid of certain new processes in differential topology. A particular application is the proof of the existence of at least $\frac{1}{2}n(n-1)$ closed geodesics on any regular manifold homeomorphic with an $(n-1)$-sphere. The latter result affords a general setting for certain results in the two-dimensional case obtained by Poincaré, Birkhoff, and certain pupils of Hadamard.

107. Professor F. R. Bamforth: *Surface transformations.*

In the first part of this paper are discussed surface transformations of period two in the neighborhood of an invariant point, and it is shown that the equations representing such a transformation, after a proper choice of coordinate system has been made, have one of the forms (i) $u_1=u$, $v_1=v$, (ii) $u_1=-u$, $v_1=-v$, or (iii) $u_1=u$, $v_1=-v$. In the second part of the paper are studied surface transformations, each of which is the product of two transformations of the type (iii) above. This discussion is made relative to a neighborhood of a point which is invariant under each of the component transformations, and treats such topics as stability, and the existence of formal and actual invariant curves through the invariant point, formal invariant series, and formal invariant integrals.

108. Professor D. V. Widder: *Singular points of functions which satisfy the partial differential equation of the linear flow of heat.*

In 1903 M. Bôcher proved that a single-valued harmonic function of two variables x, y with an isolated singularity at a point (a, b), but bounded on one side at least, must have the form $A \log [(x-a)^2+(y-b)^2]+H(x, y)$, where A is constant and $H(x, y)$ is harmonic in a neighborhood of (a, b). It has long been recognized that the source solution $(y-b)^{-1/2}e^{-(x-a)^2/[4(y-b)]}$, $y>b$, plays a rôle in the theory of the equation $\partial^2 u/\partial x^2-\partial u/\partial y=0$ analogous to that of the function $\log[(x-a)^2+(y-b)^2]$ in the theory of harmonic functions. It is the purpose of the present note to extend the analogy further by obtaining an analog of Bôcher's theorem.

109. Professor W. J. Trjitzinsky: *On indefinitely differentiable and quasi-analytic functions.*

This paper is, to an extent, based on certain results of Poincaré, Borel, de la Vallée Poussin, Carleman, and some others. The series of the form $\sum x_n f(a_n x)$, where $f(x)$ and all its derivatives are bounded as a set for all real values of x, are studied in general as well as from the point of view of their application to indefinitely differentiable functions. In particular, necessary and sufficient conditions are given (relating to the law of decrease of the x_n) in order that $\sum x_n f(nx)$ should represent a quasi-analytic function. Furthermore, several extensions are given of representations of quasi-analytic functions in terms of the initial values at a point. Finally, Borel's monogenic functions (quasi-analytic in the plane) are shown to be representable by certain double series, uniformly convergent over certain perfect subsets of their domains of definition.

110. Professor I. M. Sheffer: *Differentiation and integration of matrices, and functional equations in matrices.*

There are a number of definitions of the derivative of a matrix. The one here offered is made to depend on the canonical decomposition of a matrix: $P=\phi^{-1}P^*\phi$, where P^* is a diagonal matrix whose elements are the zeros of the characteristic equation for P. We regard P as varying only in its characteristic values, and define Q as a "proper function" if it has the form $Q=\phi^{-1}Q^*\phi$, the elements of Q being functions of those of P in a certain sense. The derivative dQ/dP is then defined as the limit of a difference quotient. This definition permits of an inverse operation, matrix integration. Further, various matrix-functional equations can be handled, among them a class of integral equations in matrices.

111. Dr. Hillel Poritsky: *On certain approximations to analytic functions of several variables.*

Let R be a given region bounded by a surface S, f an analytic function. In this paper are considered approximations to f over R by means of the solution of $\nabla^{2n+2}p_n=0$ which satisfies the boundary conditions $p_n=f$, $\nabla^2 p_n=\nabla^2 f, \cdots, \nabla^{2n}p_n=\nabla^{2n}f$ on S, $n=0, 1, \cdots$. These approximations, or rather their equivalent expansions, are related to certain other boundary value expansions previously considered (*Cauchy-Green expansions*, abstract in this Bulletin, March–April, 1929) in the same way that the expression of a harmonic function in terms of its boundary values by means of Green's function is related to its expression by means of Green's "double layer" integral. The nature of the convergence of the two expansions is different, however. For the latter expansions a sufficient condition was shown to be that f be analytic in a sufficiently large region enclosing R. A sufficient condition for the convergence of

p_n to $f(z)$, it is shown, is that the Taylor series of f be dominated by the Taylor series of an exponential Ce^A, $A=\alpha_1 x+\alpha_2 y+\alpha_3 z$, $\alpha_1^2+\alpha_2^2+\alpha_3^2<\rho^2$, where ρ is the smallest value of λ for which there exists a non-trivial solution of $\nabla^2 u+\lambda^2 u=0$ in R, $u=0$ on S. In the above inequality the constant ρ may not be replaced by any larger value.

112. Dr. Hillel Poritsky: *A note on an isoperimetric inequality.*

It is shown by means of Schwartz's inequality that the area of the sector swept out by the radius of a circular arc of length l is less than the area swept out by the radius of curvature of any other (convex) curve segment of length l, with tangent (normal) that turns through an angle equal to the angular opening of the sector, as the point of contact describes the segment.

113. Professor Georges Calugaréano: *On differential equations admitting polygenic integrals.*

In an earlier paper (Transactions of this Society, vol. 31) the author showed that some differential equations admit more general integrals than those given by Cauchy's theorem; such differential equations, designated as of class (α), admit polygenic integrals, in addition to the analytic integrals given by Cauchy's theorem. In the present paper, he furnishes a characterization of this class (α).

114. Dr. J. A. Shohat: *On Tchebycheff polynomials of best approximation.*

Let $\psi(x)$ be monotonic non-decreasing in a finite interval (a, b). If the number of its points of increase is everywhere dense in (a, b), then $P_n(x)$, the polynomial of degree $\leqq n$ minimizing $I_m=\int_a^b |f(x)-P_n(x)|^m d\psi(x)$, approaches, as $m\to\infty$, the polynomial $\pi_n(x)$ of best approximation to $f(x)$ on (a, b), and $I_m^{1/m}$ approaches, as $m\to\infty$, the best approximation $E_n(f)$. The same holds true if $\psi(x)$ has only a *finite* number $\mu(\geqq n+2)$ of points of increase $\{x_i\}$ in (a, b), with the understanding that $\pi_n(x)$ represents now the polynomial of best approximation $(=E_n(f))$ to $f(x)$ over the set of points $\{x_i\}$. This enables us to obtain very simple explicit expressions for $\pi_n(x)$ and $E_n(f)$ for the latter case, which, as we know, is of fundamental importance for the general case.

115. Dr. J. A. Shohat: *On certain inequalities for Stieltjes integrals.*

The author extends to Stieltjes integrals a determinant-transformation given by E. Fischer for ordinary definite integrals (Archiv für Mathematik und Physik, vol. 13 (1908), pp. 32–49) and shows that the Tchebycheff and Schwartz (and many similar) inequalities follow from this transformation by specifying the number and the nature of the functions involved. Also a remainder for the Tchebycheff inequality is given. The results are applied to finite sums written in the form of Stieltjes integrals. An abstract of a part of the above results appeared in the Paris Comptes Rendus (vol. 189 (1929), pp. 618–620).

116. Miss Helen M. Schlauch: *Mixed systems of linear equations and inequalities.*

In papers entitled *Systems of linear inequalities*, L. L. Dines has found a necessary and sufficient condition for the existence of solutions in systems of linear inequalities, and W. B. Carver has found a different kind of necessary and sufficient condition for the non-existence of such solutions. These theorems naturally suggest a consideration of systems containing both linear equations and linear inequalities. The present paper gives certain necessary and sufficient conditions both for the non-existence and for the existence of solutions of such systems.

117. Dr. John Williamson: *A special prepared system for two quadratics in n variables.*

From a geometric point of view it is desirable that a complete system for two quadratics in n variables should preserve intact the compound coordinates that necessarily appear. Complete systems have been determined in which these compound coordinates are decomposed into their component point or plane variables. While not finding a complete system, this paper determines a prepared system, in terms of which every concomitant of the two quadratics, if multiplied by a suitable invariant factor, may be expressed. If the n quadratic covariants of the two original quadratics are denoted symbolically by i_x^2, $i=1, 2, 3, \cdots, n$, this prepared system consists of 2^n-1 factors of the types i_x, (ijp_2), $(ijkp_3)$, $\cdots$, $(123 \cdots n-1, n)$, where the symbols i, j, k, etc., occurring in any one factor, are distinct and p_2, p_3, etc., denote compound coordinates. To obtain a prepared system, in terms of which every concomitant, not multiplied by an invariant factor, may be expressed, other more complicated factor types must be added to this system. The number of such new factor types for the cases $n=4$, 5, and 6 is 1, 8, and 52 respectively.

118. Professor L. T. Moore: *A note on the multiple factors of a binary form.*

In this paper the author makes an attempt to find transvectants of a binary form which are perfect powers of the multiple factors of the form.

119. Dr. A. A. Albert: *New results in the theory of normal division algebras.*

The author shows that in every normal division algebra A in sixteen units over R, the field of all rational numbers, there are eight integer parameters ρ, σ, $\gamma_1, \cdots, \gamma_6$, where neither ρ, σ, nor $\rho\sigma$ is a rational square. Algebra A is associative if and only if $(\gamma_5^2-\gamma_6^2\sigma\rho)=(\gamma_1^2-\gamma_2^2\rho)(\gamma_3^2-\gamma_4^2\sigma)$ and is a division algebra if and only if the ternary quadratic form $\lambda_1^2-\sigma\lambda_2^2-(\gamma_1^2-\gamma_2^2\rho)\lambda_3^2$, in the integer variables λ_1, λ_2, λ_3, is not a null form. As all ternary quadratic null forms are known, this amounts to a construction of all normal division algebras of order sixteen over R in terms of the solutions of a single diophantine equation which satisfy a single quadratic residue condition. Necessary and sufficient algebraic conditions that A be a cyclic algebra are given, and it is shown that all of the algebras constructed by Cecioni are cyclic. An algebra

A is shown to be non-cyclic if and only if two quartic forms with coefficients polynomials in ρ, σ, $\gamma_1, \cdots, \gamma_6$ are non-null forms.

120. Dr. A. A. Albert: *The non-existence of pure Riemann matrices with normal multiplication algebras of order sixteen.*

The most interesting question in the theory of Riemann matrices is that of the existence of pure Riemann matrices with non-commutative multiplication algebras. In a recent paper the author found necessary and sufficient conditions on a normal division algebra that it be a multiplication algebra of a pure Riemann matrix and gave examples of matrices whose algebras are generalized quaternion algebras and of matrices whose algebras are generalized quaternion algebras over a quadratic field. In the present paper it is shown that there exist no pure Riemann matrices whose multiplication algebras are normal division algebras over the field of all rational numbers.

121. Dr. A. A. Albert: *Integral bases of all normal quartic fields.*

All normal quartic fields, that is, fields generated by a root of a quartic with cyclic group or "Vierergruppe" are considered. By using elementary methods the author obtains *explicit formulas*, in terms of the coefficients of the quartic equations giving the fields, for a set of bases of the integers of all normal quartic fields. These formulas are comparable to the known formulas for quadratic fields and involve only elementary number-theoretic functions.

122. Dr. A. A. Albert: *A determination of the integers of all cubic fields.*

The integers of every cubic field generated by a root of an equation $x^3+b=0$ are known (see J. Sommer, *Vorlesungen über Zahlentheorie*). They have been determined by the use of certain congruences. Using these same congruences, the author obtains explicit formulas for the integers (that is, for an integral basis) of any cubic field. The reduced form $x^3+ax+b=0$ of the general cubic is used, and a set of bases involving factors of the discriminant $-4a^3-27b^2$ and a and b is obtained.

123. Professor W. L. G. Williams: *Note on the summation of homogeneous functions of n variables over all points of a modular n-space.*

We take p any prime number and n any positive integer; by "point" in a modular n-space we mean any n-tuple $(x_1, x_2, \cdots, x_n)$ in which the x's are integers, reduced mod p, and the last x(if any) which is not zero is 1. The fundamental theorem of the present paper is as follows: For all non-negative integral values of $i_1, \cdots, i_n$, $\sum x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$, summed over all the points of the modular n-space, is, for every choice of the i's, congruent, mod p, to $(-1)^{n+1}$ or zero: to $(-1)^{n+1}$ if $0<i_j\equiv 0$, mod $(p-1)$, for every value of j from 1 to n, and to zero in all other cases.

124. Professor H. B. Curry: *The foundations of combinatory logic.*

The knowledge presupposed in a logical theory ordinarily includes certain categories, of which proposition, propositional function, etc., are examples, and also certain more or less complicated modes of combination which are indicated by the use of variables. The criteria by which we conclude that an entity constructed by combining other entities belongs to this or that category are ordinarily not among the formal developments of the theory, but are the subject of intuitive reasoning. There is thus a presupposed theory of considerable complexity, so much the greater because it is essentially from this pre-logic that the paradoxes arise. The author proposes to subject this pre-logic to mathematical treatment; the resulting theory he calls combinatory logic. In the present paper a part of this program is completed, namely, the analysis of the modes of combination considered as formal processes. A set of primitive ideas, rules, and postulates is given, from which, it is shown, all the properties of the combinations, which follow from the usual representation, can be abstractly deduced. The essential primitive ideas have already been given by Schönfinkel, but he has to consider them intuitively, rather than, as here, abstractly. It is also shown that the present system is consistent with the ordinary one.

125. Professor H. B. Curry: *The universal quantifier in combinatory logic.*

This paper is a continuation of the author's program in combinatory logic. The knowledge here subjected to analysis is the following principle: if a formula is true for all values of the variables contained in it, and we substitute in it a second formula which may contain some new variables, then the new formula is true for all values of all the variables. It is shown how, if certain axioms of much simpler form be added to the list given in the preceding paper, the equivalent of this property can be proved in any case by a formal argument, which, incidentally, does not involve any use of variables. A by-product is the proof that (essentially) if the formula $(x_1, x_2, \cdots, x_m)\cdot f(x_1, x_2, \cdots, x_m) \supset g(x_1, x_2, \cdots, x_m) : \supset : (x_1, \cdots, x_m) f(x_1, \cdots, x_m) \cdot \supset \cdot (x_1, \cdots, x_m) g(x_1, \cdots, x_m)$ be assumed as axiom for $m=1$, then any particular case for any value of m can be deduced abstractly.

126. Professor Orrin Frink, Jr.: *The fourth postulates of Riesz and Hausdorff.*

The author shows by an example that not every space which satisfies the four postulates of Riesz for points of accumulation, and in which every derived set is closed, is a Hausdorff space. A paper on the same subject was read by R. G. Putnam at the meeting of the Society in New York in March, 1929 (see this Bulletin, vol. 35, p. 442).

127. Professor H. M. Gehman: *A special type of upper semi-continuous collection.*

The main result of this paper is that the topology of a plane S in which a given bounded continuum K is considered as an element of an upper semi-continuous collection of continua, the other elements of which are the points of $S-K$, is the same as that of a surface consisting of a plane and a contracting sequence of spheres, each of which is tangent to the plane at the same point of the plane. By means of this result a number of theorems concerning accessibility may be proved as corollaries of previously known theorems.

128. Dr. J. H. Roberts (National Research Fellow): *A characterization of the anchor ring by internal properties.*

Leo Zippin has shown that a bounded continuous curve in n-space without a cut point which satisfies the Janiszewski-Mullikin theorem is topologically equivalent to a sphere. In the present paper it is shown that the following two conditions are necessary and sufficient that a closed point set be an anchor ring: (1) the set M contains two mutually exclusive simple closed curves, J_1 and J_2, such that $M-J_i (i=1, 2)$ satisfies the Janiszewski-Mullikin theorem, and (2) every subset of M which lies on any arc whatsoever lies on an arc which is a subset of M, and (3) if J is any simple closed curve of M such that $M-J$ is connected, then $M-J$ has the Jordan curve property.

129. Dr. J. H. Roberts (National Research Fellow): *A note on a theorem of Mazurkiewicz and Straszewicz.*

Mazurkiewicz and Straszewicz have proved (Fundamenta Mathematicae, vol. 9) that if H and K are two closed sets neither separating 3-space and such that neither H nor K is interlaceable but $H\cdot K$ is, then $H+K$ separates 3-space. The following stronger theorem, for which their proof suffices, seems to be of sufficient importance to deserve mention: If H and K are continua such that $H\cdot K$ is interlaceable, and every simple closed curve interlaced with $H\cdot K$ contains at least one point of both H and K, then $H+K$ separates 3-space.

130. Dr. N. E. Rutt (National Research Fellow): *Prime ends and indecomposable continua.*

This paper is devoted to an investigation of relations between the two following types of plane bounded continua: (1) those which are the boundary of some domain at least one prime end of which contains its entire boundary, (2) those which are indecomposable. The chief conclusions of the paper are contained in the two following theorems. Theorem I: If the bounded plane continuum Δ, boundary of the plane domain δ, is indecomposable, there exists a prime end of δ containing Δ. Theorem II: If the boundary of the plane domain δ is a bounded continuum Δ, and there exists a prime end of δ containing Δ, then Δ is either indecomposable or the sum of two indecomposable continua.

131. Professor W. A. Wilson: *A property of unbounded continua, with applications.*

It is shown that, if $\{X_i\}$ is a sequence of continua lying in a metric space satisfying the Bolzano-Weierstrass condition and either each continuum is unbounded or the diameters increase indefinitely with i, then the upper closed limit, if not void, is an unbounded continuum or a closed set each component of which is an unbounded continuum. This theorem is used first to derive various theorems regarding unbounded irreducible continua, in particular a sufficient condition for a continuum containing two points to contain a continuum irreducible between these points. With the aid of these the author's theorem on regular plane frontiers (this Bulletin, vol. 34, p. 86) is extended to unbounded frontier sets.

132. Professor R. G. Putnam: *Note on equivalence of certain properties of abstract sets.*

In a paper *Sur l'équivalence de trois propriétés* (Fundamenta Mathematicae, vol. 2), Sierpinski proves the equivalence, in classes (L), of three properties and also the equivalence of two others. In this note it is shown that these properties are equivalent in certain classes (V) and theorems concerning classes (V) are obtained from this result.

133. Professor O. D. Kellogg: *An unsolved problem in potential theory.*

As is now well known, the Dirichlet problem cannot be solved for every region and every continuous boundary function. A generalized function of Green may be set up for an arbitrary region by the method of Harnack (*Grundlagen der Theorie des logarithmischen Potentiales*, Leipzig, 1887, pp. 116–121), and the points of the boundary at which this function approaches 0 are called *regular* boundary points. All others are *exceptional*. The author gives a method, which includes as special cases a number of classical methods, for attaching to any region and any continuous boundary values, a function which is harmonic and bounded in the region, and which approaches the boundary values at every regular boundary point. The unsolved problem of uniqueness is this: Does there exist a second such function? This problem has been answered in the negative for logarithmic potentials (Comptes Rendus, vol. 187 (1928), p. 526), but the method used is not available in space of three or more dimensions. It is highly desirable that definite results be obtained in the latter case, not only as a completion of a theory, but because of the large number of theorems which would follow (see, for instance, Vasilesco, in an article soon to appear in the Journal des Sciences Mathématiques, also Bouligand, Annales de la Société Polonaise de Mathématique, 1926). The author sketches a number of methods of approach to the problem, and developes several related theorems. In addition to the citations above, the following references will be found to bear on the subject: Acta Universitatis, Szeged, vol. 4 (1928), pp. 1–5; American Journal, vol. 51 (1929), pp. 515–526; Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 322 ff., pp. 335–337.

134. Professor E. P. Lane: *Integral surfaces of pairs of partial differential equations of the third order.*

In a linear space of n dimensions a point has $n+1$ projective homogeneous coordinates $x^{(1)}, \cdots, x^{(n+1)}$. If these coordinates x are functions of two independent variables u, v, the locus of the point x, as u, v vary, is a surface. If the coordinates x satisfy a linear homogeneous partial differential equation of the third order in the single dependent variable x and the two independent variables u, v, the surface is said to be an integral surface of the equation. This paper studies integral surfaces of a pair of such equations. A study of a certain system of covariantly defined triples of one-parameter families of curves on an integral surface, and, more particularly, a consideration of the so-called singular triples, make it possible to reduce the system of equations to one or the other of six canonical forms. Each canonical form is characterized by a projective geometric description of its integral surfaces. The integrability conditions are discussed, and some geometric results are obtained.

135. Professor A. D. Michal: *The analog of the projective connection in function space.*

In an earlier paper the author obtained the function space analog of Weyl's projective curvature tensor. The present paper continues the study of projective functional invariants. In particular, the function space projective connection is introduced and its law of transformation is studied.

136. Professor C. N. Moore: *On the nature of the composite numbers in an arithmetic progression.*

One of the classical results of the theory of numbers is Dirichlet's well known theorem that an arithmetic progression in which the first term and the common difference are relatively prime contains an infinite number of primes. It is natural to put the question as to whether or not anything definite may be said regarding the nature of the composite numbers in such a progression. In the present paper this question is answered in the affirmative by showing that the progression contains an infinite number of numbers which are the products of two primes, an infinite number which are the products of three primes, and so on. The proof is based on certain properties of multiple Dirichlet's series.

137. Dr. Marie J. Weiss (National Research Fellow): *On groups defined by* $A^q=1$, $B^{-1}AB=A^x$, $B^Q=A^e$.

The purpose of this note is to investigate some of the properties of the solvable, non-abelian groups generated by $A^q=1$, $B^{-1}AB=A^x$, $B^Q=A^e$, where B^Q is the least power of B found in the group $\{A\}$. A normalization of the generators and some necessary conditions for the simple isomorphism of two such groups are given.

138. Professor W. L. Ayres: *On subsets of locally connected continua as Borel sets.*

It is shown that if $\epsilon>0$ and H is the set of all points p of a locally connected continuum M such that $M-p$ has at least two components of diameter $\geqq\epsilon$, then H is a closed point set. As a corollary we have a new proof that the set of all cut points of M is an F_σ. The author proves that the set of end points of M is an $F_{\delta\sigma}$ but not necessarily an F_σ, and that the set of all points of M that belong to some simple closed curve of M is a $G_{\delta\sigma}$ but not necessarily a G_δ. This answers two questions of G. T. Whyburn (Mathematische Annalen, vol. 102, p. 319).

139. Dr. W. T. Reid (National Research Fellow): *Generalized Green's matrices for compatible systems of differential equations.*

In this paper it is shown that generalized Green's matrices exist for a general compatible system consisting of n ordinary linear differential equations of the first order: $y_i' = \sum_{j=1}^n A_{ij}(x) y_j$, $(i=1, 2, \cdots, n)$, where the functions $A_{ij}(x)$, $(i, j=1, 2, \cdots, n)$, are Lebesgue summable, and n linearly independent boundary conditions involving the values of the functions $y_i(x)$, $(i=1, 2, \cdots, n)$, at two points. For a treatment of the analogous problem for a single differential equation of the nth order, see W. W. Elliott, American Journal of Mathematics, vol. 50 (1928), p. 243.

140. Professor G. A. Miller: *Inverse correspondences in automorphisms of abelian groups.*

This paper considers automorphisms of abelian groups and proves the following theorems: A necessary and sufficient condition that the operators of a given subgroup of an abelian group may correspond to their inverses in an automorphism of this group in which no other operator corresponds to its inverse is that this subgroup involves all the characteristic operators of the group. In particular, if an abelian group contains no characteristic operator besides the identity then it is possible to establish an automorphism of this group in which the operators of an arbitrary subgroup correspond to their inverses while no other operator of the group has this property. A necessary and sufficient condition that an automorphism of the cyclic group of order p^m, p being an odd prime number, is of order $2p^\beta$ is that $p^{m-\beta}$ of its operators correspond to their inverses under this automorphism.

141. Dr. E. Kathryn Wyant: *The ideals in the algebra of generalized quaternions over the field of rational numbers.*

In this paper the author considers the algebra which is composed of the set of elements $\xi = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2, and x_3 are rational numbers, where $i^2 = -a$, $j^2 = -b$, and $ij = -ji = k$, and where a and b are odd and prime to each other. A system of integers in this algebra has been found, its discriminant has been computed, and in it the resolution of rational primes into two-sided prime ideals has been determined. Basal units of the integers and ideals have been found.

142. Professor J. S. Turner: *Some identities connected with Fermat's last theorem.*

In this paper, the equation $a^n + b^n = c^n$, $(a, b, c, n$ positive integers, $n > 2)$, is transformed by various trigonometrical formulas. Attempts are made to find the highest powers of 2 and n contained in the non-zero members of the resulting equations. In this way many interesting identities are discovered.

143. Professor T. A. Pierce: *Matrices whose characteristic equations are cyclic.*

If the characteristic equation of a matrix is cyclic this equation will have its full complement of matric roots. An isomorphism exists between the algebraic and matric roots. Several consequences of this isomorphism are given.

144. Dr. J. J. L. Hinrichsen: *On the problem of n bodies.*

In the Chicago Colloquium Lectures of 1920, Birkhoff established certain results concerning motions in the problem of three bodies. He showed if the sum of the kinetic and potential energies is positive or zero and the constant of total angular momentum is not zero, at least two if not all three of the mutual distances increase indefinitely as the time increases or decreases, and that the same result is true in case the total energy is negative if the motion is such that for some instant all three bodies approach sufficiently near together. In conclusion he stated that the argument used admits of certain generalizations. In the present paper such a generalization is considered. The corresponding results are established, under the assumption that none of the bodies ever collide, not only for a system of n bodies attracting one another according to the Newtonian inverse square law but also for the case of n bodies attracting one another with a force inversely proportional to the $(d+1)$st power of the mutual distances where $0<d<2$, and which reduces to the Newtonian case for $d=1$.

145. Mr. E. W. Anderson: *Limits of approximate solutions of the torsion problem.*

Ritz's method of approximation for a "boundary value problem," such as $\nabla^2\psi=0$, and $\psi=\psi_0(x, y)$ on the boundary, gives no indication as to what degree of approximation the method provides, except to provide an upper limit. Trefftz's method gives a lower limit, so that both methods give upper and lower bounds. In this paper an accurate solution is given for the torsion of a prism bounded by arcs of confocal parabolas. Approximate solutions sufficiently accurate for technical practice, and rather readily obtained, are given as border solutions of the true result.

146. Professor H. P. Evans: *A two-dimensional boundary-value problem for the transmission of alternating currents through a semi-infinite heterogeneous conducting medium.*

We consider an infinitely long conductor parallel to the surface of a semi-infinite conducting medium. This medium is supposed to consist of an upper stratum of uniform conductivity σ_1 and having everywhere else a uniform conductivity σ_2. The conductor is supposed to carry an alternating current for which the medium forms the return path. The problem is that of determining the field vectors throughout space, subject to the condition that the frequency is sufficiently low so that displacement currents may be neglected. This problem has been partially solved by Haberland (Zeitschrift für Angewandte Mathematik und Mechanik, vol. 6 (1926)), who used approximate boundary conditions which are reasonable in case the stratum is sufficiently thin. The present treatment utilizes exact boundary conditions and permits any thickness for the stratum. The boundary conditions are formulated for a function which satisfies the wave equation and may be expressed in terms of an infinite integral from which the field vectors are derivable. This integral is expanded into a convergent series of simpler integrals which are given physical interpretation.

147. Professor H. A. Meyer: *On certain inequalities with applications in actuarial theory.*

Inequalities bearing a marked relation to the first and second theorems of mean value, giving in many cases rather close bounds to the integrals or finite sums considered, have been derived by Tchebycheff, Hölder, Jensen, Meidell and Steffensen. These inequalities, when applied to certain problems in life contingencies, provide a means of determining upper and lower limits of otherwise complicated and unmanageable expressions. These inequalities were studied with the object of generalization and removal of restrictions on the functions involved, together with the extensions of their actuarial applications. The principal results obtained are as follows: Steffensen's inequality is extended and from it Meidell's inequality is derived as well as another inequality similar to that of Meidell. Tchebycheff's inequality is also generalized somewhat. The extent of the actuarial application of these inequalities was rather gratifying.

148. Professor Henry Schultz: *The standard error of a forecast from a curve.*

Gauss' formula for the standard error (s.e.) of any function of a number of independent quantities whose s.e.'s are known may be used to obtain the s.e. of a curve in terms of the s.e.'s of its parameters. The resulting expression is a function of the independent variables, which are assumed to be free from errors of observation. If in this equation we assign to the independent variables values lying beyond the range of the observations, we obtain the s.e. of any extrapolation, or forecast, from the curve. The s.e.'s of the straight line, parabola, cubic, etc., and of all planes, all increase indefinitely as we increase the range of the extrapolations. The s.e. of the population logistic $y=b/(e^{-ax}+c)$ increases as the curve is extrapolated toward its upper asymptote, and approaches a constant value as $x\rightarrow\infty$. It decreases as the curve is extrapolated toward its lower asymptote and approaches 0 as $x\rightarrow-\infty$. When the logistic is fitted to the population of the United States, it points to a maximum population, which will be practically attained by 2100, of 196 millions. By accepted methods, the s.e. of this figure is only ± 0.8 millions. By the method of this paper, it is ± 10.5 millions.

149. Professor H. L. Rietz: *On certain properties of frequency distributions whose variates are obtained by linear fractional transformation of the variates of a given distribution.*

In a former paper (Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 817–820), the author gave some properties of frequency distributions of powers and roots of the variates of a given distribution. The present paper considers certain properties of the distribution of variates obtained by applying a linear fractional transformation to each variate of a given positive unimodal frequency distribution with a range of variates from -1 to $+1$.

150. Professor P. R. Rider: *On the probability associated with a given range.*

If m is the mean and σ the standard deviation of a normally distributed variate x, then, as is well known, the probability that a value of x selected at random will lie within the range $m \pm 3\sigma$ is 0.997. If $\bar{x}$ and s are the mean and the standard deviation respectively of a sample, the expected or average probability associated with the range $\bar{x} \pm 3s$ will be different from the probability associated with the range $m \pm 3\sigma$. Shewhart (Journal of Forestry, vol. 26 (1928), pp. 601–7) obtained experimentally for the average probability for samples of four, associated with the range $\bar{x} \pm 3s$, the values 0.90 for a normal universe, 0.91 for a rectangular universe, and 0.91 for a triangular universe. This paper considers the question theoretically and derives the value 0.92 for this average probability in the case of samples of four from a discrete rectangular universe. The frequency distribution of probabilities associated with the range $x \pm 3s$ also is given.

151. Professor Harold Hotelling: *The consistency and ultimate distribution of optimum statistics.*

For estimating the unknown parameters of a frequency distribution various quantities are used, for example, means, medians, correlation coefficients, known generically as *statistics.* An optimum statistic is defined as the value of a parameter which makes the probability of the observed data a maximum. In this paper it is proved that, unlike some other statistics in common use, an optimum statistic has in large samples, under certain conditions involving first derivatives, the following properties: It approaches the true value of the parameter. Its distribution in samples of n tends to the normal form as n increases. The variance, multiplied by n, tends to the reciprocal of the mean value of the square of the simultaneous optimum estimates of two or more parameters; the matrix of variances and covariances is n^{-1} times the inverse of the matrix of the mean values of the products of the first derivatives.

152. Professor G. R. Davies: *An analysis of frequency distributions.*

This paper was presented by invitation of the Program Committee of Section K of the American Association for the Advancement of Science.

153. Professor G. E. Raynor: *On the Dirichlet-Neumann problem.*

The paper deals briefly with the problem of finding a function harmonic in a region R with given boundary values and given values of its normal derivatives on the boundary. A uniqueness theorem is proved and an existence theorem for certain conditions on the boundary values and the boundary is given.

154. Dr. H. S. Wall: *On the Padé approximants associated with a positive definite power series.* (Second Communication.)

Denote by (1) $S_0, S_1, S_2, \cdots$ the principal diagonal file and the successive parallel files to the right of it respectively, in the Padé table for $P(z)$, a positive definite power series. In a preliminary report (this Bulletin, vol. 35 (1929),

p. 451), the author discussed the convergence of (2) S_0, S_2, $S_4, \cdots$ and like files below the principal diagonal. Concerning (3) S_1, S_3, $S_5, \cdots$, and like files below the principal diagonal, it is shown that when $P(z)$ is indeterminate either all converge or else all diverge. A necessary and sufficient condition for convergence is found. The limits of successive files (1) are distinct except that sets S_{2k-1}, S_{2k}, S_{2k+1} of not more than three files may have equal limits. When $P(z)$ is determinate and just the first $k+1$ files (2) are equal to $f(z)$, those of the first k files (3) equal $f(z)$, while subsequent files are of the same character as when $P(z)$ is indeterminate. On the question of "extending" $P(z)$ to the left, results are found which are somewhat analogous to the author's results for a Stieltjes series.

155. Professor D. L. Holl: *Some properties of* $\nabla^4\psi=0$, *when expressed in curvilinear coordinates.*

When the complex transformation $W=f(z)$ is applied to $\nabla^4\psi=0$, the form of the equation is not preserved, and the form of the function ψ in the new region cannot be inferred from its form in the original region. In this paper, those transformations are determined which preserve the form of $\nabla^4\psi$, also those which yield $\nabla^4(h\psi)$, where $h=|f'(z)|$. Other special cases are discussed and applications cited.

156. Mr. H. C. Carter: *Relation of Maschke's symbolic method to the tensor theory.*

In Maschke's symbolic method for the study of invariants of quadratic differential forms, such a form G is expressed symbolically as $G=\sum g_{ij}dx^i dx^j=(\sum f_i dx^i)^2=(\sum \phi_i dx^i)^2$, etc. Thus the coefficients g_{ij} are represented symbolically as products $f_i f_j$ or $\phi_i\phi_j$, etc. In developing the theory it is known that symbolic invariants occur: that is, expressions which are formally invariant but which contain symbols f_i, ϕ_i, etc., each occurring just once as a factor of each term and which therefore cannot be interpreted in terms of the g_{ij}'s as indicated above. In certain cases a vector interpretation of these symbolic invariants is possible but the more general invariants of this type cannot be interpreted in this way. In the present paper Maschke's method is generalized and it is shown how the coefficients in the general symbolic invariants may be interpreted as tensors or as quantities related to tensors. This method is used to build up complicated tensors from more simple tensors. Other applications are also given.

157. Professor A. D. Michal: *The differential geometry of a continuous infinitude of contravariant functional vectors.*

In this paper the author develops the theory of a continuous infinitude of contravariant functional vectors. The theory is then applied to the infinitely many-dimensional group manifold of infinite groups of functional transformations with arbitrary functions.

158. Professors A. D. Michal and C. T. Bumer: *On dynamical*

systems with n degrees of freedom subject to hysteresis effects of the Fredholm type.

If hysteresis effects of the Fredholm type are introduced in the classical theory of dynamical systems with n degrees of freedom, the equations of motion become integro-differential equations. In this paper the authors study the properties of such curious dynamical systems and obtain functional invariants of their equations of motion.

159. Professor A. D. Michal: *On some remarkable theorems on geodesic coordinates of order r in n-dimensional differential geometries.*

The author studies in this paper a class of geodesic coordinates y^i for which the first $r-1$ generalized affine connections $\Gamma^i_{\alpha\beta\cdots\gamma}$ vanish at the origin when evaluated in the preferred system. The theorem is proved that any two such geodesic coordinate systems $\bar{y}^i$ and y^i with the same origin are related by a transformation for which $\partial^\rho \bar{y}^i/(\partial y^\alpha \cdots \partial y^\beta)(\rho=2, 3, \cdots, r+1)$ vanish at the origin. The theories of tensor extensions, normal tensors, replacement theorems, etc., are then developed under lightened hypotheses on the affine connection.

160. Professor A. D. Michal: *Concerning algebro-functional groups of transformations and the theory of projective functional tensors.*

In this paper a study is made of groups of functional transformations on the composite range of a function $y(x)$ and a variable u. In particular, this theory is used in developing a harmonious theory of projective functional tensors in function space.

161. Professor A. D. Michal: *Dynamical systems with infinite degrees of freedom and their integral invariants.*

In this paper the author studies dynamical systems whose kinetic energies are given by functionals of the form $g_{\alpha\beta}[y](\partial y^\alpha/\partial t)\ (\partial y^\beta/\partial t)+g_\alpha[y](\partial y^\alpha/\partial t)^2$. The generalized Lagrangian and Hamiltonian systems are then deduced. The main part of the paper is concerned with the development of a theory of functional invariants of integro-differential equations and its application to the theory of the generalized Hamiltonian systems.

162. Professor M. M. Slotnick: *A note on a projective invariant of a conjugate net.*

Corresponding tangential directions to the first and minus first Laplace transforms of a conjugate net intersect the axis of the net in two ranges of points which are in projective correspondence. The invariant of this projectivity is a projective invariant of the net. This invariant also has its dual.

163. Professor F. E. Wood: *The projective equivalence of two nets of conics.*

If ϕ_i, $\psi_i (i=1, 2, 3)$ are quadratic forms in three variables, then two nets of conics $\lambda_1\phi_1+\lambda_2\phi_2+\lambda_3\phi_3=0$, $\lambda_1\psi_1+\lambda_2\psi_2+\lambda_3\psi_3=0$ will be projectively equivalent if and only if there is a projective transformation which carries the first net into the second. Conditions both necessary and sufficient that two nets be projectively equivalent are obtained in this paper, which from this point of view supplements a paper by Jordan, *Réduction d'un réseau de formes quadratiques ou bilinéaires*, (Journal de Mathematiques, (6), vol. 2 (1906) pp. 403–38), in which all the special cases are fully considered. In this paper the general case is treated by a method different from that used by Jordan, who does not obtain a necessary condition, involving the coefficients of the forms ϕ_i, ψ_i, that two nets shall be projectively equivalent; indeed it seems impossible to obtain such a condition by his method. In this paper certain new theorems regarding nets of conics and related cubics are obtained in a preliminary section.

164. Professor J. S. Turner: *A condition for the concurrence of three common chords of three conics, taken in pairs.*

Let the given conics be C_1, C_2, C_3. Let S_1, S_2, S_3, be three conics having double contact (all internal, or all external) with pairs of C_1, C_2, C_3. The equations of the latter can be written $S_2-u_1^2\equiv S_3-v_1^2=0$, $S_3-u_2^2\equiv S_1-v_2^2=0$, $S_3-u_3^2\equiv n(S_3-v_3^2)=0$. Then if $n=1$, six common chords are concurrent by threes in four points. The condition is satisfied if S_1, S_2, S_3 are circles. The results also hold for sphero-conics. Acknowledgment is due to J. L. Coolidge for the method of proving concurrence.

165. Dr. Nola L. Anderson and Professor Louis Ingold: *Invariant normals to a space S_n contained in a function space.*

This paper continues Dr. Anderson's study of normals to given spaces. A space S_n of n dimensions is defined by a function $f(t; x^1, x^2, \cdots, x^n)=f(t, x)$, $(\alpha\leqq t\leqq\beta)$, with x^i as coordinates of the space. The functions $f_i=\partial f/\partial x_i$ are called tangents to the parameter curves. Any function $N(t; x)$ such that $\int_\alpha^\beta f_i(t; x)N(t; x)dt=0$ is a normal to S_n. Normals in the form $N=\sum P^{ij}(x)f_{ij}+\sum Q^i(x)f_i$, where $f_{ij}=\partial^2 f/\partial x^i\partial x^j$, are *first normals* to S_k. The functions $N_{ij}=f_{(ij)}$ (the parentheses indicate covariant differentiation) constitute a symmetric covariant set of first normals. $\sum A^{ij}N_{ij}$ denotes an invariant normal if A^{ij} denotes a tensor depending on the x^i only. By applying Maschke's symbolic method tensors A^{ij} are derived from the functions f_i and their derivatives alone, which therefore do not depend on special loci or on special tensor or vector fields of S_n. Each gives an invariant normal; for example, $N=\sum g^{ij}N_{ij}$, where g^{ij} is obtained from the fundamental quantities $g_{ij}=\int_\alpha^\beta f_{ij}fdt$ in the usual manner. This is the analog of the well known mean normal to a variety V_n.

166. Professor A. H. Blue: *On the structure of sets of points of classes one, two, and three.*

The Lebesgue classification of sets of points is based upon their representation in terms of open sets and closed sets. The notion of the type of a set is introduced to make a distinction between those sets which have the same open class but are of different closed classes, and between those sets which have the same closed class but are of different open classes. One of the principal ressult is the theorem: If two sets, which are products of open sets, are everywhere dense in a perfect set, they have a common point. The structures of sets of types (1, 1), (1, 2), (2, 1), (2, 2), and (3, 2) are discussed.

167. Professor Dorothy McCoy: *The complete existential theory of eight fundamental properties of topological spaces.*

A system $(P; K)$, called a topological space, is composed of an aggregate P of elements p and a relation K among the subaggregates of P. Eight properties of systems $(P; K)$ and their negatives are studied in detail. These include the four properties of F. Riesz (Atti del 4 Congresso Internazionale dei Matematici, Roma, vol. 2, page 18; 1910), together with closure of derived sets, separability, and the properties of being perfect and connected. If the cardinal number of the class P is unrestricted it is shown by examples that but one relation of implication can exist among the eight properties studied. The cases of a class P with a finite or enumerably infinite number of elements are exhaustively investigated. It is found that all cases which can occur on a space of a finite number of elements greater than one can be obtained on spaces of four elements, but not on spaces of two or three elements. Interesting results are obtained for spaces which are topologically homogeneous.

168. Professor M. G. Gaba: *A set of axioms in terms of points, ordinary and ideal.*

In vol. 11 of the Transactions of this Society, F. W. Owens has a paper entitled *The introduction of ideal elements and a new definition of projective n-space.* He based his work on eleven axioms. The first eight are essentially the first eight of Veblen's axioms published in the 1904 Transactions of this Society in his memoir *A system of axioms for geometry.* Owens' ninth axiom is a closure axiom and the tenth and eleventh are the Desargues theorem and its converse. However, before enunciating the last two axioms, it was necessary for him to give lengthy and involved definitions of *in pencil* and *in range.* In the present paper the results of Owens' paper are obtained by axioms that are far more simple in statement, and the use of but one short definition instead of the seven that he used.

169. Dr. Morris Marden (National Research Fellow): *On the zeros of certain rational functions.*

Thanks to a suggestion from Professor Pólya, the author is able to give in this note a "best" zero-free region for the rational functions $\sum_{j=1}^{n}\alpha_j/f_j(z)$ and $\sum_{j=1}^{n}\alpha_j f_j(z)$, where every $f_j(z)$ is a polynomial of actual degree m and has all of its roots in a given convex domain, and where the α_j are complex numbers such that for all j, $k+\pi<k+2\pi-\gamma\leqq \text{arc } \alpha_j\leqq k+2\pi$. The principal theorem of the note thus generalizes some theorems not only of the author's recent article on *Zero-free regions of linear partial fractions* (this Bulletin, vol. 35 (1929), pp. 363–370), but also of Nagy's article on *Die Lage der Wurzeln von*

linearen Verknüpfungen algebraischer Gleichungen (Acta Universitatis Hungaricae Francisco-Josephinae, vol. 1 (1923), p. 127). The former results are included in the special case $m=1$, $\gamma=\pi/2$; the latter in the special case $\gamma=0$.

170.* Professor E. P. Lane: *Integral surfaces of triads of partial differential equations of the third order.*

The purpose of this paper is to study the integral surfaces of a system of three linear homogeneous partial differential equations of the third order. The first problem is to determine what canonical forms are possible for such a system. It is found that any such triad of equations can be reduced to one or the other of *three* canonical forms, each of which is characterized by a different projective geometric description of its integral surfaces. For each of the canonical forms there are certain integrability conditions, which are discussed. And there are some geometric theorems which are immediate consequences of the analytic results obtained. The methods used in this paper are quite similar to those employed in a recent paper by the same author on *Integral surfaces of pairs of partial differential equations of the third order.* (Received January 12, 1930.)

171. Professor H. E. Arnold: *Notes on the Jonquière quartic.*

Consider a Jonquière quartic and the pencil of binary quartics cut out by the lines through any given point of the plane. In this paper several properties of quartics belonging to this pencil are derived, and its relationship to the fundamental involution, to which all line sections of the curve are apolar, is discussed. (Received January 25, 1930.)

172. Professor J. E. Donahue: *On the geometry of the second derivative of a polygenic function.*

In Kasner's article on *The second derivative of a polygenic function* (Transactions of this Society, vol. 30, pp. 803–818) there appears a cardioid associated with a limaçon. In this paper an investigation is made of the exact relations of the cardioid and the limaçon. It is based partially on the following theorem: Any three parallel lines in a plane determine the size of a cardioid tangent to them and the angle which the cuspidal tangent makes with any one of these parallels, except for sign. The constant defining the size of the cardioid, the point of tangency of the cardioid and the limaçon, and the orientation of the cuspidal tangent are determined in terms of the constants in the equation of the limaçon, the angle formed by one of the enveloping lines, and the radius of the limaçon associated with it. (Received February 1, 1930.)

173. Dr. Lulu Hofman: *Plane transformations preserving centers of gravity.*

A plane transformation $u=\phi(x, y)$, $v=\psi(x, y)$ of the z-plane into the w-plane is considered, the planes being covered with mass of density 1 in z and $1/|\phi\psi|_{xy}$ in w, so that the masses of two corresponding finite areas A_z and

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* The papers beyond this point are to be read at meetings of which the reports have not yet been published.

A_w are equal. We denote by $z_g(x_g, y_g)$ and $w_g(u_g, v_g)$ the centers of gravity of A_z and A_w respectively, and we study the following problems: (a) given a set of areas A_z, to find all transformations $\phi(x, y)$, $\psi(x,y)$ such that z_g and w_g correspond in the transformation $u_g=\phi(x_g, y_g)$, $v_g=\psi(x_g, y_g)$; (b) given a transformation $\phi(x, y)$, $\psi(x, y)$, to find all areas A_z such that z_g and w_g correspond. Problem (a) has been solved completely if the given A_z are (i) the ∞^3 circles, (ii) the ∞^4 rectangles parallel to a fixed direction, or (iii) all areas of z. In case (i) ϕ and ψ are harmonic, in case (ii) they are of the form $\phi=\alpha+\beta x+\gamma y+\delta xy+\epsilon(x^2-y^2)$, $\psi=\alpha'+\beta'x+\gamma'y+\delta'xy+\epsilon'(x^2-y^2)$, where $\delta'/\epsilon'=\delta/\epsilon$, and in case (iii) they determine an affine transformation. Problem (b) is discussed when the transformation is of the form of case (ii) above, with unrestricted coefficients. Among the A_z are then contained all squares and all areas with rotational symmetry such that the smallest rotation admissible is $\neq\pi/2$ and $\neq\pi$. Additional problems are outlined. (Received January 30, 1930.)

174. Dr. A. A. Albert: *A necessary and sufficient condition for the non-equivalence of any two rational generalized quaternion division algebras.*

The problem of the non-equivalence of any two generalized quaternion division algebras over the field of all rational numbers is considered. It is shown that every pair of algebras may be taken in a canonical form in which, if $1, i, j, ij$ are a basis of the general algebra, the squares of i in the two algebras are the same rational number ρ. It is then shown that the two algebras are equivalent, when taken in this form, if and only if the squares γ and δ of the j's satisfy a relation $\delta=(\eta_1^2-\eta_2^2\rho)\gamma$ for rational η_1 and η_2. Necessary and sufficient diophantine conditions on ρ, γ, and δ are given that the above relation be true, a complete solution of the problem. (Received January 6, 1930.)

175. Professor Oystein Ore: *On hypergeometric functions of several variables.*

A hypergeometric function of two variables $F(x, y)=\sum_{n,m=0}^{\infty} a_{n,m}x^ny^m$ is defined by the fact that the quotients $a_{n+1,m}/a_{n,m}=R_1(n, m)$, $a_{n,m+1}/a_{n,m}=R_2(n, m)$ are *fixed* rational functions of n and m. Horn, Kampé de Fériet, and Birkeland have pointed out that the functions R_1 and R_2 cannot be arbitrary. In this paper the explicit form of the possible coefficients is determined. (Received January 17, 1930.)

176. Mr. Samuel Borofsky: *Linear homogeneous differential equations with Dirichlet series as coefficients.*

A differential equation $d^ny/dx^n+P_1(x)d^{n-1}y/dx^{n-1}+\cdots+P_n(x)y=0$ is considered, where each P is the sum of two Dirichlet series, one with zero and negative exponents and one with positive exponents, both absolutely convergent in a right half-plane. It is shown that there exists a linearly independent set of solutions $y_i=e^{r_ix}[\phi_{i0}(x)+x\phi_{i1}(x)+\cdots+x^{p_i}\phi_{ip_i}(x)]$, $i=1, 2, \cdots, n$, where r_i is some complex number and the ϕ's are Dirichlet series with non-positive exponents absolutely convergent in a right half-plane, if and only if no positive exponents appear in the P's. This includes the classical Fuchs theory for the so-called "regular singular point" for the case when the P's are Laurent series. (Received February 1, 1930.)

177. Dr. J. H. Roberts (National Research Fellow): *A point set characterization of 2-dimensional closed manifolds.*

In her thesis (Mathematische Annalen, vol. 98 (1927)) Miss Gawehn gives a characterization of closed 2-dimensional manifolds in terms of point set notions. The present paper offers the following characterization: In order that a closed and compact point set M lying in a metric space S be a closed 2-dimensional manifold it is necessary and sufficient that (1) if T is any *arc set* (that is, T is closed, and every component t of T which contains more than one point is an arc of which at most the end points are limit points of $T-t$) lying in M, then M contains an arc which contains T, and (2) for some $k(k\geqq 0)$ M contains $2k$ simple closed curves, $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_k, \beta_k$ such that (a) $\alpha_i \cdot \beta_i$ is at most one point and $(\alpha_i+\beta_i)\cdot(\alpha_j+\beta_j)=0(i\neq j;\ i, j\leqq k)$, and (b) if, for each i, γ_i denotes either α_i or β_i, and $K=M-\sum_{i=1}^{k}\gamma_i$, then K is connected, but every simple closed curve in K separates K. As a special result the integer k is taken to be 1 and the manifolds so obtained are distinguished from each other. (Received February 1, 1930.)

178. Professor J. E. Donahue: *On the region covered by a certain polynomial.*

There is determined in this paper the boundary of the region of the complex plane covered by the polynomial $z=a_0+a_1t_1+a_2t_2+a_3t_1t_2$, in which the coefficients are complex and $t_1=e^{i\theta_1}$ and $t=e^{i\theta_2}$, θ_1 and θ_2 being independent variables. (Received February 1, 1930.)

179. Professors Orrin Frink and P. A. Smith: *Irreducible non-planar graphs.*

One of the results of this paper is a simple necessary and sufficient condition that an arbitrary linear graph be mappable on a plane. (Received February 10, 1930.)

180. Dr. A. A. Albert: *A construction of all non-commutative rational division algebras of order eight.*

All rational generalized quaternion algebras and all rational cyclic algebras of order sixteen which are division algebras have been constructed. Also a type of cyclic division algebra of order nine has been constructed. In the present paper the problem of a construction of all algebras which are generalized quaternion algebras over a quadratic field $R(u)$ is considered. It is shown that when the algebra is not a direct product of $R(u)$ and a generalized quaternion algebra then it is a division algebra. Diophantine necessary and sufficient conditions that this be so are also given. The remaining case, in which A is a direct product of $R(u)$ and a normal division algebra of order four, is treated and explicit diophantine conditions on the multiplication constants of A are given which insure that A is a division algebra. (Received January 6, 1930.)

181. Professor G. C. Evans and Mr. W. G. Smiley, Jr.: *The first variation of a functional.*

The authors obtain sufficient conditions for the representation of the first variation of a functional in terms of a Stieltjes and a Lebesgue integral respectively, the difference in the two representations corresponding to a difference in the nature of the continuity of the functional in the two cases. (Received December 30, 1929.)

182. Dr. Lincoln LaPaz: *Geometries in which straight lines are shortest.*

Geometries in which length of arc is defined by an integral of type $L=\int_{x_1}^{x_2}f(x, y, y')dx$ and in which straight lines are shortest were first studied by Hamel, who employed the method of Darboux to determine the form of the integrand function f. In the present paper this problem is treated by the use of theorems established by the writer in an earlier article (see abstract (15), this Bulletin, vol. 35 (1929), p. 761). Conditions are obtained on the integrand function $f(x, y, y')$ sufficient to insure that the straight line segment joining arbitrary but fixed points (x_1, y_1) and (x_2, y_2), $x_2 \neq x_1$, shall render the integral L a proper absolute minimum with respect to all other admissible arcs $y=y(x)$ joining the two given points. It is shown that Hamel did not obtain the most general geometry for which straight lines are shortest. (Received January 25, 1930.)

183. Professor G. T. Whyburn: *A continuum every subcontinuum of which separates the plane.*

In this paper there is constructed a bounded plane continuum M such that (1) M is the common boundary of two domains; (2) every subcontinuum of M separates the plane; (3) M is topologically contained in every one of its subcontinua; (4) M contains no uncountable collection of mutually exclusive continua and therefore no indecomposable continuum; obviously it contains no arc; (5) M admits of upper semi-continuous decomposition into elements (continua or points), all save a countable number of which are points and with respect to which M is a simple closed curve; and (6) M contains two continua that are not homeomorphic with each other. It is proved, furthermore, that any plane continuum having property (2) contains a bounded continuum having properties (1) and (2) and that every bounded continuum having (1) and (2) also has properties (4), (5), and (6). From these facts it follows that no plane continuum which separates the plane can be homeomorphic with every one of its subcontinua. (Received January 15, 1930.)

184. Professor G. T. Whyburn: *Completely partitionable connected sets.*

A connected point set will be called completely partitionable provided that every two points of M may be separated in M by some third point of M. Any one of the following conditions characterize completely partionable connected sets M in a separable metric space: (1) that there exist a biunivalued and continuous transformation T defined on M such that $T(M)$ is a subset of an acyclic continuous curve; (2) that every two points of M may be joined in M by a unique relatively closed irreducibly connected set and that components of the complement of every connected relatively closed subset of M be open in M; (3) that M be regular in the sense that every point P of M is the divisor of a family of open subsets of M the M-boundary of each of which is finite, and that, for each point P of M and each component C of $M-P$, P be a point of order 1, in this sense, of $C+P$. A connected set M is topologically contained in an acyclic continuous curve if and only if it is completely partitionable and the components of the complement of any compact subset of M form a contracting sequence. (Received January 15, 1930.)

185. Professor G. T. Whyburn: *On non-separated cuttings of connected point sets.*

A collection Q of subsets of a connected point set M will be called non-separated if the elements of Q are mutually exclusive and no element of Q separates in M any two points belonging to a single element of Q. It is shown that for every uncountable non-separated collection G_0 of cuttings of a connected set M in a separable metric space, a subcollection G of G_0 exists containing all save a countable number of the elements of G_0 and such that each element g of G is an irreducible cutting of M whose complement in M has just two components, g is saturated in M relative to G, i.e., g is separated in M from each point of $M-g$ by some element of G, and g is an element of order two of M relative to G in the sense that it is the divisor of a family of open subsets of M whose M-boundary is a subset of two elements of G. Furthermore, if G is any such collection, M may be decomposed into a collection H of elements containing G as a subcollection and with respect to which M is a completely partitionable connected set. (Received January 15, 1930.)

186. Professor G. T. Wayburn: *Concerning certain continuous curves in the plane and connected and connected im kleinen point sets in general.*

It is shown that every plane continuous curve every subcontinuum of which is a continuous curve is a rational curve in the sense of Menger, i.e., every point can be ϵ-separated in the curve by a countable set of points. Also, if A and B are any two points of a compact, connected, and connected im kleinen set M in a metric space, there exists an uncountable non-separated collection Q of cuttings of M between A and B; thus M can be decomposed, with the aid of Q, into a non-separated collection G of elements with respect to which M is a simple continuous arc. Other decompositions by this means of such sets M are studied. It is also shown, in answer to a question raised by the author (Fundamenta Mathematicae, vol. 13, p. 50) that every open subset of any connected and connected im kleinen point set M contains a completely irreducible cutting of M. (Received January 15, 1930.)

187. Mr. Hassler Whitney: *A normal form for maps.*

It is shown that, given any map on the surface of a sphere of n regions, $n \geqq 3$, in which (A_1) the boundary of each region is a single closed curve without multiple point, (B) all the vertices are triple, (A_2) no two regions taken together form a multiply connected region, (A_3) no three regions taken together form a multiply connected region, a closed curve may be drawn which passes through each region once and only once, and passes through no vertex. Therefore the map may be represented in the dual form as an n-sided polygon, the interior and exterior having been subdivided by diagonals into triangles. The vertices of the polygon represent the regions of the map, and any two vertices of the polygon connected by a line represent two regions with a common boundary. The polygon itself represents the closed curve drawn through the map. (Received January 28, 1930.)

188. Professor George Rutledge: *A necessary and sufficient condition for convergence of a sequence of Lagrange polynomials.*

In a recent number of the Transactions of this Society (vol. 31, pp. 807–820) the writer presents the sequence of polynomials determined by $2n+1$ evenly spaced points with n increasing indefinitely in a form in which the coefficients of x, x^2, $\cdots$ appear as double series. In this paper it is shown that convergence of the coefficients of x and x^2 is sufficient for convergence of the sequence of polynomials. It is also necessary. The sequence of polynomials determined by a given function $f(x)$ is equivalent to the sequence of partial sums of an expansion of the function in Stirling's series. (Received February 1, 1930.)

189. Dr. T. C. Benton: *On certain types of surfaces in three-dimensional euclidean space.*

A definition of an edge point of a surface is given in terms of the separating set for the point. Also a definition is given for a flat surface. These two properties are shown to be invariants under homeomorphism. A type of accessibility in three dimensions is discussed in which a point p on a surface S is called conically accessible from an unknotted simple closed curve C not on S if it is possible to construct an equicontinuous, self-compact set of arcs each having a point of C as one end point and p as the other end point and such that every point of C is an end point of such an arc and no point of these arcs except p is on S. Certain properties of flat surfaces, with and without edge points, which are conically accessible from every unknotted simple closed curve in the complementary domains of the surface are then proved. (Received February 5, 1930.)

190. Mr. L. A. Dye: *Involutorial transformations of order N with an $(N$-$1)$-fold line.*

Montesano (Lombardo Istituto Rendiconti, (2), vol. 21 (1888)) has given a brief synthetic discussion of involutorial transformations of order N with an $(N-1)$-fold line. In this paper two methods of analytic treatment are developed, and applied to the cases $N=3$ to 7. It is also shown that the transformations can be mapped on S_3 and are therefore rational. (Received February 10, 1930.)

191. Professor L. E. Dickson: *Construction of division algebras.*

This paper obtains a decided simplification of the theory of the associativity conditions presented in the author's paper *New division algebras* (Transactions of this Society, April, 1926). It then obtains final results in new general cases, which were too complicated by the earlier methods. The paper will appear in the Transactions of this Society. (Received February 17, 1930.)

192. Professor J. L. Walsh: *On the overconvergence of sequences of polynomials of best approximation.*

This paper proves, by means of a single theorem on the degree of approximation to an analytic function, many well known results and a number of new ones on the overconvergence of sequences of polynomials of best approximation to an analytic function and to a harmonic function. (Received February 3, 1930.)

193. Professor Edward Kasner: *The third characteristic property of the derivative of a polygenic function.*

The derivatives $\gamma = dw/dz$ of any polygenic function $w = \phi(x, y) + i\psi(x, y)$ may be regarded as establishing a correspondence between the direction elements (x, y, y') of the z-plane and the points of the γ-plane. How is such a correspondence characterized among all possible element-to-point transformations? The author shows that in addition to the two properties given in his earlier papers (see Science, vol. 66 (1927), p. 581; Proceedings of the National Academy, vol. 13 (1928), p. 75), namely (1) elements at a point go into points of a circle, (2) corresponding arcs on the circle and angles at the point are in the ratio $-2:1$, we must adjoin as a further property exactly how the clocks thus defined are in correspondence with the points. If we define the clock by its central vector A and its principal phase vector B, a certain displacement corresponds to Δz and another displacement to the conjugate value $\Delta \bar{z}$. The vectorial difference $\Delta A - \Delta \overline{B}$ then defines what we call the *associated affinity.* Property (3) then states that this affinity is a direct similitude. The three properties are necessary and sufficient for polygenic functions. If the function is monogenic, (3) reduces to ordinary conformality. (Received February 5, 1930.)

194. Dr. T. H. Gronwall: *Rotational symmetry in the characteristic functions of the Schrödinger wave equation.*

In this paper, certain consequences of the invariance of the Schrödinger wave equation under the group of rotations, which were proved by Weyl by means of the theory of group representations (*Gruppentheorie und Quantenmechanik*, Leipzig, 1928, chapter 3) are obtained by a direct and elementary method. (Received February 22, 1930.)

195. Professor G. Y. Rainich: *Unification of force and matter components.*

Preceding work showing the close relationship between the hydrodynamic and electromagnetic stress tensors suggests that the quantities in which these are quadratic, namely electric and magnetic forces and momentum vector of matter, must also be related, as components of the same higher unity. This paper introduces an antisymmetric tensor in five-space, six of whose components are electromagnetic and the remaining four represent matter; its matrix is obtained by bordering that of the electromagnetic force tensor by momentum components of matter. To justify this we let indices go from one to five in the usual expression for the electromagnetic tensor and obtain a five-dimensional symmetric tensor whose four-dimensional part is the hydrodynamic plus the electromagnetic stress tensor (other components give four-dimensional ponderomotive force). Letting indices in Maxwell's equations for empty space go from one to five we obtain Maxwell's equations for space filled with matter, the continuity equation of matter, and equations expressing proportionality between the momentum and the potential vectors, a relation obtained recently from another point of view by Lanczos. Equating to zero the divergence of the five-dimensional stress tensor, we obtain equations of motion. (Received March 5, 1930.)

196. Professor R. L. Wilder: *A converse of the theorem regarding the separation of E_3 by a closed two-dimensional manifold of genus p.*

The following theorem is established: Let K be a closed and bounded point set in E_3 (euclidean space of three dimensions) such that (1) the Betti number (mod 2) $R^0(E_3-K)\geqq 2$, and the Betti number $R^1(E_3-K)$ is finite; (2) if D is a domain complementary to K, then the O-chains of D are uniformly homologous to zero in D and every point of K is a limit point of D. Then K is a closed two-dimensional manifold of genus $p=[R^1(E_3-K)-1]/2$. This theorem contains the converse of the Jordan-Brouwer separation theorem in E_3 as a special case (that is, where $R^1(E_3-K)=1$; see this Bulletin, vol. 35, p. 771, abstract No. 49), and the conditions which it embodies are all known to be necessary. (Received March 6, 1930.)

197. Mr. C. W. Mendel: *The dimensionality of a certain linear space in the projective differential geometry of a variety in hyperspace.*

The element E_r at a point P of a curve C in space S_n is the totality of P and the osculating linear spaces of C at P up to and including the one of dimensions r. The osculating space $S(k, r)$ with respect to an element E_r at a point P of a curve C on a variety V_m in space S_n is the linear space of least dimensions containing the osculating spaces S_k at the point P of every curve on the variety V_m through the point P, having at P the element E_r in common. These definitions are due to Segre and Bompiani. In certain special cases these writers found the dimensionalities of the osculants they had defined but the general formula seems not to have been previously announced. This paper shows by induction that the dimensionality of the space $S(k, r)$ is $(r+1)C_{m+p-1,\, m}+(c+1)C_{m+p-1,\, p-1}$, where the integers p, c are defined by the relations $k=p(r+1)+c$, $0\leqq c\leqq r$, and $C_{m+p-1,\, m}$ is the binomial coefficient $(m+p-1)!/(p-1)!m!$. In the special case for which $m=2$, the variety V_m is a surface and the dimensionality of $S(k, r)$ is given by the formula $[k(k+r+3)+c(r-c-1)]/2(r+1)$ where c is again the least residue of k modulus $r+1$. (Received February 10, 1930.)

198. Professor Arnold Emch: *On algebraic surfaces which are invariant in a certain class of finite collineation groups.*

This paper deals with the properties of symmetric surfaces of the form $\sum x_i^n x_k^n = 0$, $i, k = 1, 2, 3, 4$; $i \neq k$, which are invariant under finite collineation groups of order $24n^3$, or $8(24n^3)$, according as n is odd or even. Certain configurations of conics and lines which exist on the surface are investigated. As an example, a sextic surface is given with four triple points such that a sphere through these cuts the surface in 6 circles which lie in the faces of a cube such that each circle passes through the vertices of the corresponding square. (Received February 26, 1930.)

199. Mr. W. O. Pennell: *A generalization of Fourier series.*

This paper describes an infinite series $S(x)$ of sines and cosines which represents for all values of x, positive or negative, a function as follows: $S(x) = b^n(x - na)$, $(na < x < (n+1)a)$, where n takes on positive and negative integral values including zero, b is any real constant; positive or negative, and $f(x)$ is a function defined in the interval $0 < x < a$. If unity is substituted for b the series reduces to the classical Fourier series for $f(x)$, $0 < x < a$. A number of expansions are worked out and approximation curves are shown and discussed. The series was discussed by methods of the operational calculus and these methods are used in the paper. (Received February 27, 1930.)

200. Dr. L. M. Blumenthal: *Groups of permutable functions.*

Volterra has given a method (Sopra le funzioni permutabili, Rendiconti Reale Accademia dei Lincei, 1910, p. 428) by which the group of functions permutable with a given function of the first order may be obtained. The method, involving the solution of an integro-differential equation, is important as an existence theorem—not as a means of obtaining an explicit expression for the group sought. In this paper an explicit formula is developed giving the group of functions permutable with a given function $F(x, y) = A(x)B(y)$ of the first order. Apart from the group of functions permutable with unity (the group of closed cycles) no other general formula of this type seems to be in the literature. The method consists in showing that the canonical form (Volterra & Peres, Lecons sur la Composition, p. 36) of $A(x)B(y)$ is unity. The canonical transformation preserves composition and permutability. Thus the case is reduced to the case of functions permutable with unity. In this way it is found that the group of functions permutable with the function $A(x)B(y)$ of first order is given by the expression $A(z)B(y)\theta\left[\int_z^y A(\xi)B(\xi)\,d\xi\right]$ where θ is an arbitrary function. (Received March 1, 1930.)

201. Professor R. E. Langer: *On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order.*

The solutions of an equation $u'' + p(x)u' + \{\lambda r(x) + q(x)\}u = 0$, with large complex λ, depend upon this parameter. The asymptotic formulas for this dependence are known for the case in which x remains on an interval on which $r(x) \geqq \alpha$(or $r(x)) \leqq -\alpha$) for some positive constant α. The methods used to obtain these formulas do not permit of simple extension to the case in which $r(x)$ has a zero. In different intervals on opposite sides of such a zero the forms representing a specific solution are different, and the relation between them has not hitherto been determined. This paper deals with the solutions of the equation on an x interval which includes a zero of $r(x)$. The character of the solutions in a neighborhood of this zero is determined, and asymptotic formulas are derived for the solutions in the remaining parts of the interval. As an application a familiar special equation, solved by the Bessel function $J_\rho(\rho e^x,)$ is considered. The general theory yields in this case formulas representing $J_\rho(\rho \text{ sech } \alpha)$ and $J_\rho(\rho \sec \beta)$ respectively for large, intermediate, and small values of sech α or sec β. These results are in accord with the familiar formulas. Certain features of difference are, however, noted. (Received March 4, 1930.)

202. Professor O. J. Peterson: *On the rational plane quintic curve with three cusps.*

It is shown that if the plane be divided into three sets of triangles, four triangles in each set, by the six lines determined by four singular points of which either one or three are cusps, then the two remaining singular points lie in triangles of the same set. The quintics are classified according to the triangles in which these two points are located. Another classification of rational quintics is on the basis of the sequences of the double points. It is shown that if the rational quintic has precisely three cusps, ten different sequences are possible. As is to be expected, the positions and the sequences of the double points are not independent, and it is interesting to observe the correspondence between them. (Received March 6, 1930.)

203. Mr. G. W. Starcher: *A solution of a simple functional equation as a basis for readily obtaining the theory of elliptic functions.*

The purpose of this note is to show, by considering certain solutions of simple functional equations, how one may obtain expressions for each of the four Theta functions, both as infinite products and as Fourier series, and thus have a basis for the theory of elliptic functions. (Received March 7, 1930.)

204. Dr. M. A. Basoco: *On certain finite sums of binomial coefficients and gamma functions.*

In this note the author obtains the value of certain finite sums involving factorials or Gamma functions, which are the immediate consequence of the orthogonality property of certain systems of polynomials. The results obtained are equivalent to the orthogonality property in the sense that they imply and are implied by this property. The systems of polynomials here considered are those bearing the names of Jacobi, Hermite, Laguerre, Legendre and Tschebycheff. This paper will appear in a forthcoming number of the American Mathematical Monthly. (Received March 7, 1930.)

205. Dr. M. A. Basoco: *On Appell's decomposition of a doubly periodic function of the third kind.*

In a series of papers in the Annales de l'École Normale (1884 to 1888) and in Acta Mathematica (1920), Appell has studied the problem of expanding the doubly periodic functions of the third kind in trigonometric series. His fundamental contribution in this connection is the discovery of a certain function $\chi_m(x, y)$ of two complex variables, which in his theory plays a role analogous to that of the Zeta function in Hermite's decomposition of an elliptic function into simple elements. In this note, the results of Appell are established for the case where the function to be expanded has more poles than zeros in a fundamental period cell, *without asssuming a priori the existence and properties of the function* $\chi_m(x, y)$. This is accomplished by applying a well known theorem relative to the expansion of a class of singly periodic functions in a series of cotangents. In this manner, one is led in a very natural way to the consideration of the function $\chi_m(x, y)$ as the fundamental element of the theory and to the proof of Appell's theorem for the case under consideration. (Received March 7, 1930.)

206. Professor W. E. Milne: *The numerical determination of characteristic numbers.*

This paper presents a method for the numerical calculation of characteristic numbers or energy levels in wave mechanics for cases where the wave equation contains, or can be reduced so as to contain, a single space variable. The procedure consists in the numerical integration of an auxiliary differential equation for several chosen values of the energy, after which the characteristic values are obtained by interpolation. Incidentally the process also furnishes the characteristic solution or wave function with little additional labor. The method is one of considerable generality, and is capable of giving any preassigned degree of accuracy. (Received March 7, 1930.)

207. Mr. S. A. Pollock: *An apparatus for the exhibition and study of skew curves.*

By means of ruled surfaces and surfaces made of light, obtained by passing parallel rays of light through a directrix curve, skew curves appear in full perspective together with their singularities. Their relations to the surfaces on which they lie are thus exhibited in three dimensions. By moving either surface, the variation of the curve of intersection is plainly observed. Plane projections are easily obtained by indirect observation, either through a mirror or by actual photographing. The apparatus will be exhibited and several examples of skew curves will be shown. (Received March 7, 1930.)

208. Dr. B. W. Jones (National Research Fellow): *Certain quinary forms related to the sum of five squares.*

In this paper the number of solutions of $n=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2$ where $a_i=1$, 2, or 4 is shown, with the exception of three such forms, to be expressible in terms of φ, where $\varphi(n)$ is the number of representations of n as the sum of five squares. Also for $a_i=1$, 2, 4 or 8 the number of solutions is expressed in terms of φ and representations by the forms (1, 1, 1, 2, 4) and (1, 2, 4, 4, 8). (Received March 7, 1930.)

209. Professor A. D. Michal: *An operation that generates scalar differential invariants from tensors.*

This paper is concerned with scalar extensions of tensors. The operations involved are based on a system of independent contravariant vectors $\xi^i{}_{(\alpha)}$,

($\alpha = 1, 2, \cdots, n$), and a symmetric affine connection $\Gamma_{\alpha\beta}{}^{i}$. A number of preliminary theorems are proved in connection with a class of geodesic coordinates of order r that is distinct from the class of geodesic coordinates of order r previously studied by the author. The paper closes with some applications to simply transitive continuous groups. (Received March 8, 1930.)

210. Dr. H. T. Engstrom (National Research Fellow): *On generalizing trigonometric identities in arithmetical paraphrasing.*

When an elliptic function is represented in more than one way by trigonometric series certain identities in cosines or in sines are obtained. To obtain arithmetic results from such identities the trigonometric functions are replaced by arbitrary functions of the same parity. By use of properties of cyclotomic fields the author gives a simple algebraic proof for this replacement principle with some generalizations. (Received March 8, 1930.)

211. Dr. Gordon Pall (National Research Fellow): *Simultaneous quadratic and linear representation.*

The problem of representing two numbers a and b, one in a quadratic, the other in a linear form, is shown to be equivalent to that of representing a certain combination of a and b in a single quadratic form, generally with some congruential restriction on the variables. Several especially simple relations are obtained, and the case of sums of squares is treated for further application. (Received March 8, 1930.)

212. Dr. Clifford Bell: *On a pair of simultaneous second order differential equations.*

In his paper on self-adjoint ordinary differential equations of the fourth order (American Journal of Mathematics, vol. 52 (1930), p. 171–196), W. M. Whyburn considers the following pair of second order differential equations, $y''+p(x)y=q(x)z$, $z''+p(x)z=r(x)y$, and gives a rather complete treatment for the cases $qr<0$ and $qr>0$ on the given interval. The present paper investigates cases that arise when q and r vanish. Some of the results and proofs of the paper are similar to those of Whyburn's paper, while others differ considerably and involve new methods of proof. (Received March 8, 1930.)

213. Professor W. M. Whyburn: *On systems of cyclically related differential equations.*

The system of first order differential equations $dy_i/dx=\sum_{k=1}^{k=n} A_k(x)y_{i+k}$, ($i=1, \cdots, n$, $y_{i+n}\equiv y_i$), where $A_1, \cdots, A_n$ are summable functions, is studied. The solutions are shown to have many properties in common with the sine and cosine functions. The cases $n\leqq 4$ are solved and methods of solution are given for higher order cases. The methods used are variations of those used by the author in a study of nonlinear second order systems (see Transactions of this Society, vol. 30 (1928), pp. 848–854). (Received March 10, 1930.)

214. Professor B. A. Bernstein and Mr. Nemo Debely: *A practical method for computing the modular representations of finite operations and relations.*

In previous papers Professor Bernstein has developed a general theory whereby any n-ary operation or any m-adic relation in a finite class of elements can be represented arithmetically. The theory rests on the fact that for K a class consisting of the p numbers $0, 1, \cdots, p-1$, p prime, any K-closing *unary* operation $f(x)$ can be represented by a polynomial of the form $a_0+a_1x+\cdots+a_{p-1}x^{p-1}$, modulo p, where the range of x is $0, 1, \cdots, p-1$, and the a's are all numbers of this range. Though this polynomial can always be found by the method of the general theory, the necessary work is very great when p is large, involving, as it does, the evaluation of p determinants, modulo p, each of order $p-1$. The present paper develops a method of obtaining the fundamental polynomial with extreme ease. (Received March 10, 1930.)

215. Professor Morgan Ward: *The algebra of recurring series.*

This paper develops a general method for obtaining relations between the solutions of a linear difference equation of order three whose characteristic equation is irreducible. The method is to set up a one-to-one correspondence between the cubic field associated with the characteristic equation and a field of matrices of order three whose elements are solutions of the difference equation. By equating corresponding elements in properly chosen matric identities an indefinite number of formulas may be obtained. The method is applicable to the general linear difference equation of order n provided that its characteristic equation is irreducible. The paper has been accepted for publication in the Annals of Mathematics. (Received March 10, 1930.)

216. Professor Morgan Ward: *Postulates for the inverse operations in a group.*

This paper defines a binary operation over a set of marks with the property that the set forms a group with respect to one of the inverses of the operation. (Received March 10, 1930.)