# MENGER ON THEORY OF DIMENSIONS

*Dimensionstheorie.* By Karl Menger. Leipzig and Berlin, B. G. Teubner, 1928. iv+319 pp.

There is an important phase in the development of modern point set theoretical geometry which has been closely associated with the concept of dimensionality,—we refer to the attempt to create precise mathematical meaning for the simple geometric spaces of our intuition in terms of primitive non-arithmetical concepts. That the idea of dimensionality should have come into play and itself have been studied and made precise is indeed natural, since the curves, surfaces, and solids of our experience furnish the very basis for our intuitive ideas of dimensionality. The simple arithmetic definition of dimensionality, however, as the number of parameters required to define a space, while useful in ordinary geometry, was of course entirely inadequate when the more abstract spaces came into consideration, and it became highly desirable that dimensionality be relieved of its arithmetical vesture and be based on the inner structure of space itself.

Consider the spaces of our experience. What non-arithmetic relations among them are intuitively certain? The following immediately suggest themselves: a solid can be separated into several parts by one or more surfaces, a surface by curves and a curve by points. It was Poincaré who in 1912 suggested that precisely this type of phenomenon might lead to a satisfactory non-arithmetic definition of dimensionality,—a definition by recurrence. A space may be called $n$-dimensional, he suggested, if it can be separated into several parts by means of continua of $n-1$ dimensions.

Although the Poincaré definition was far from satisfactory either in precision or in content, it must be regarded as of the highest historical importance, since it indicated the possibilities of definition by recurrence, and was moreover essentially topological. These virtues were recognized by Brouwer, who in 1913 slightly modified the content of the definition and stated it in terms of the topologically precise notions of separation and connectedness; as a basis for recurrence, a zero-dimensional space was defined to be one which contained no continuum as subset. Brouwer showed that this "natural" definition of dimensionality satisfied the formal requirement of yielding the number $n$ when applied to a cartesian $S_n$.

The dimensionality definition has since undergone further modification. If dimensionality was to be studied per se, it was of course desirable to arrive at a definition which would give rise to a theory of the highest generality and simplicity. To this end the basis for recurrence was altered, and what was of more significance, the concept of dimensionality as a local property was introduced. In its modern form the definition is as follows: a space is at most $n$-dimensional if each point is contained in an arbitrarily small neighborhood with a boundary of dimensionality at most $n-1$; $(-1)$-dimensional spaces are null spaces. A space which fails to be at most $(n-1)$-dimensional is at least $n$-dimensional. A space is $n$-dimensional if it is at most and at least $n$-dimensional. Finally, an obvious modification yields the definition of dimensionality at a point.

An equivalent to this definition was introduced some eight years ago by the youthful Russian mathematician Paul Urysohn, who developed his theory in one of the most brilliant memoirs of recent years (posthumously published). The definition in the form given above was introduced independently at about the same time by Professor Karl Menger of Vienna and developed in a sequence of shorter papers characterized by their elegance and generality. The essentials of the dimensionality theory, which has by now attained a considerable perfection through the recent writings of Menger, Hurewicz, Alexandroff and others, have been developed with admirable clarity and completeness in a recently published book by Professor Menger. Let us review briefly some of the notable results of the theory as it is presented by the author.

A subset of a given space may itself be regarded as a space. If each of the subsets $M_1, M_2, \cdots$, is $n$-dimensional, is the set $\sum M_i$ $n$-dimensional? The answer is yes provided the $M_i$ are closed. That the situation is entirely different in the general case is seen from Urysohn's remarkable theorem that an $n$-dimensional set is the sum of $n+1$, but not of fewer than $n+1$, 0-dimensional sets. On account of its obvious importance, the summation theorem for closed sets is developed by the author at the very outset. There is included an extremely elegant proof by Hurewicz which yields at the same time the Urysohn theorem. The polish of the modern methods is strikingly revealed when this proof is compared with the original proof of Urysohn.

In 1911 Lebesgue pointed out the following property of euclidean $n$-space: if a bounded region is covered by a finite number of closed sets of sufficiently small diameter, there must exist points which are common to $n+1$ of the covering sets. This remarkable property of ordinary space holds with suitable modifications for abstract $n$-dimensional spaces. Certain striking generalizations, together with a sort of converse which states that covering sets with intersection properties analagous to those of the (closed) $n$-cells of an $n$-complex, can always be chosen,—are among the author's own achievements. For reasons which we shall later indicate, the importance of this group of theorems can hardly be overestimated.

A significant feature of the theory as it stands at present is the relative unimportance of the assumption of compactness. The extension of the greater part of the theory to the more general separable metric spaces,—made possible by recent improvements in methods, furnishes further justification for the modern definition of dimensionality. The permanence, in this sense, of the theorems of dimensionality is analogous, as the author points out, to the permanence of the laws of arithmetic on the extension of a number domain.

There are, however, certain interesting exceptions. If a space $R$ is $n$-dimensional, it must certainly be $n$-dimensional at certain of its points. Now if $R^n$ represents the totality of such points, what is the dimension of $R^n$? If we assume that $R$ is compact, the answer is that $R^n$ is $n$-dimensional *at each of its points*. Quite different is the situation if $R$ is not compact. In fact we have actually an example, due to Sierpiński, of a one-dimensional space which is one-dimensional at only a denumerable set of points, and these points constitute a zero-dimensional set by the summation theorem. The author has succeeded in demonstrating that $R^n$ as a space can not be of fewer than $n-1$ dimensions. Whether or not $R^n$ may be less than $(n-1)$-dimensional at certain of its points remains one of the important unsolved problems in the structure analysis of separable spaces.

The cartesian spaces $S_n$ are of course $n$-dimensional in the sense under consideration; were this not the case, the definition of dimensionality would fail to satisfy the most fundamental requirement. Since dimensionality is clearly a topological invariant, we have here a proof of Brouwer's "classical" theorem concerning the topological non-equivalence of $S_p$ and $S_q$ where $p \neq q$. It is this theorem which imparts validity to the definition of $n$-dimensional complex in the sense of combinatorial analysis situs.

We see then that one aspect of the problem of determining completely the relations between abstract spaces and number spaces of a given dimensionality is solved. Another aspect, concerning which there is little known as yet, we may characterize as the problem of the introduction of coordinate systems into point-set theoretical geometry. This problem is of extreme importance, since it is just here that a fusion between combinatorial analysis situs and point-set theoretical analysis situs is beginning to take place.

The few facts in this connection which are known are highly interesting and suggestive. We have for example the fact already referred to concerning the complex-like structure of an arbitrary space of finite dimensionality. There is also the recent result due to the author, and destined, it seems, to be of considerable importance, that every $n$-dimensional space $R$ is homeomorphic to a subspace of a cartesian $S_{2n+1}$. That this theorem holds when $R$ is a complex is obvious from the remark that two $S_n$'s in $S_{2n+1}$ fail in general to intersect. It is this fact together with the complex-like structure of $R$, which furnishes a guiding principle for a proof of the general theorem. The question as to whether or not $S_{2n+1}$ is the space of lowest dimensionality in which $R$ can be immersed, is still one of the interesting unsolved problems.

We mention finally the theorem of Alexandroff that if $R$ is a closed $n$-dimensional space immersed in a euclidean space, there exists an $n$-dimensional approximating complex into which $R$ can be carried by means of a singular continuous deformation in which each point moves an arbitrarily small distance; such an approximating complex can not be of fewer than $n$ dimensions. Thus the permanence of dimensionality under a distortion depends on the magnitude of the distortion. It is essentially this fact which constitutes the germ of Brouwer's original proof of the "classical" invariance theorem.

It is impossible to compare Menger's book with the original sources without admiring the thoroughness and elegance with which he has embraced the essentials of the dimensionality theory as it stands today into a single volume of some three hundred pages. Nor can it be said that this has necessitated undue condensation. One may complain perhaps that the historical notes at the end of each section, while quite complete, avoid in their conciseness certain questions of emphasis. But as for the subject matter, there is no feeling of compression; on the contrary the author frequently pauses to suggest problems, explain difficulties and guiding principles, and emphasize values. For some theorems there are developed several distinct proofs, each bringing to light new aspects of the theory and new points of view. The essentials of the point set theory of separable spaces are developed at an early stage so that one may read the entire volume without outside reference. The material is well arranged and the printing is excellent, with exceptionally few errors. The book constitutes, in short, a notable presentation of an important chapter in modern mathematics.

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## TONELLI ON TRIGONOMETRIC SERIES

*Serie Trigonometriche.* By Leonida Tonelli. Bologna, Nicola Zanichelli, 1928. viii+523 pp.

The extent of the existing literature on the theory of trigonometric series is tremendous and keeps increasing very rapidly. During the quarter of a century after 1900 the theory has made remarkable progress and it would not be an exaggeration to say that now it is of equal and fundamental importance for all branches of the modern mathematics including the theory of numbers on the one hand, and mathematical physics on the other. In spite of, or perhaps because of this, there is practically no place in the literature where an adequate account of the theory is given, except for the second volume of Hobson's *Theory of Functions* and two excellent but short reports by M. Plancherel (L'Enseignement Mathématique, vol. 24 (1925), and by E. Hilb and M. Riesz (Encyklopädie der mathematischen Wissenschaften, vol. II, $3_2$, 1924).

Under such circumstances the publication, by a mathematician of Tonelli's rank, of a large volume devoted exclusively to the theory of trigonometric series must be considered as a significant event, even if it does not represent a step toward the solution of the difficult problem of creating an all-inclusive treatise on trigonometric series.

The work under review originated as a course of lectures on trigonometric series delivered at the University of Bologna in 1924–1925 with a view "to expounding in a systematic manner the classical results, together with more recent investigations on these series." The author found it more convenient to abandon the usual order of treatment of trigonometric series. The book begins with a study of general trigonometric series, in order "to reveal at once the properties which are common to all such series," and subsequently passes to the discussion of the special properties of Fourier series. An advantage of such a treatment lies, according to the author's opinion, in the fact that "the theory of the general trigonometric series can be presented in a form essentially elementary in character, while the theory of Fourier series, for its complete development, requires speculations of a more advanced nature." No space is given to the "beautiful investiga-