The truth of the final statement of the theorem emerges when to k are assigned successively the values 1, 2, $\cdots$, $m-1$, and when it is recalled that a sufficient condition that the manifold defined by the equations (A) has no singular points in R is that the matrix M_m is of rank m at all points of the manifold in R.

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ON THE FOUNDATIONS OF GENERAL INFINITESIMAL GEOMETRY*

BY HERMANN WEYL

In connection with a seminar on infinitesimal geometry in Princeton, in which I took part, it seemed desirable to clarify the relations between the work of the Princeton school and that of Cartan.

With a group $\mathfrak{G}$ of transformations in m variables ξ is associated, in accordance with Klein's Erlanger Program, a homogeneous or plane space $\mathfrak{R}$ of the kind $\mathfrak{G}$; a point of $\mathfrak{R}$ is represented by a set of values of the "coordinates" ξ^α and figures which go into each other on subjecting the coordinates to a transformation of $\mathfrak{G}$ are to be considered as fully equivalent. The transformations of $\mathfrak{G}$ give at the same time the transition between two allowable "normal" coordinate systems in $\mathfrak{R}$. If we have two spaces $\mathfrak{R}$, $\mathfrak{R}'$ of the kind $\mathfrak{G}$ and set up a definite normal coordinate system in each of them, then such a transformation can be interpreted as an isomorphic representation of $\mathfrak{R}$ on $\mathfrak{R}'$. $\mathfrak{G}$ is assumed to be transitive.

Cartan† developed a general scheme of infinitesimal geometry in which Klein's notions were applied to the tangent plane and not to the n-dimensional manifold M itself. The

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object of his work can be briefly described as follows. There is associated with each point P of M an m-dimensional plane space of the kind $\mathfrak{G}$. To the transition from P to a neighboring point P', that is, to the line element PP', corresponds an isomorphic representation of T_P on $T_{P'}$, the "displacement" $T_P \rightarrow T_{P'}$ or briefly $\overline{PP'}$. On this basis the introduction of a general concept of curvature is possible: if we displace the tangent plane T_P in P along a curve L on M which leads back to P the tangent plane returns in a new position or orientation. The final position is obtained from the original by a certain isomorphic representation of T_P onto itself, and this we call the "curvature along L."

For the purpose of analytic formulation we refer M to coordinates x^i and introduce a normal coordinate system ξ^α in each tangent plane T_P. Let the components of $\overline{PP'}$ be dx^i. We assume that the ξ^α of all the T_P can be so chosen that the displacement PP' is an infinitesimal isomorphic representation of the same order of magnitude as the dx^i. A radical specialization is introduced by the further assumption that this displacement depends linearly on $\overline{PP'}$, i.e. that the consecutive application of the displacements PP' and $P'P''$ shall yield the displacement PP''. The displacement from P to all neighboring points P' is then expressed by the formula

$$(1) \qquad\qquad d\xi^\alpha = -\sum_i u_i^\alpha(\xi)dx^i.$$

The curvature along the infinitesimal parallelogram formed by line elements dx and δx, which consequently has as components

$$(\Delta x)^{ik} = dx^i\delta x^k - \delta x^i dx^k,$$

is given by

$$\Delta\xi^\alpha = R_{ik}^{\alpha}(\xi)dx^i\delta x^k = \tfrac{1}{2}R_{ik}^{\alpha}(\xi)(\Delta x)^{ik},$$

$$R_{ik}^{\alpha} = \left(\frac{\partial u_k^\alpha}{\partial x^i} - \frac{\partial u_i^\alpha}{\partial x^k}\right) + \left(\frac{\partial u_k^\alpha}{\partial \xi^\rho}u_i^\rho - \frac{\partial u_i^\alpha}{\partial \xi^\rho}u_k^\rho\right).$$

In the foregoing the tangent plane is not tied up with the manifold; in order to justify this designation and hold to the

idea of a tangent plane we must now imbed it into the manifold. The first step in this process consists in taking a definite point O of T_P as center which shall, by definition, cover the point P on M (imbedment of 0th order). This leads to a restriction in the choice of a normal coordinate system ξ on T_P; because of the transitivity of $\mathfrak{G}$ it can and shall be so chosen that the coordinates ξ vanish in the center. The group $\mathfrak{G}$ is restricted to the subgroup $\mathfrak{G}_0$ of all representations of $\mathfrak{G}$ which leave the center O invariant. On displacing the tangent plane along a closed curve L, O goes over into a point O^* whose deviation from O characterizes the "torsion along L." $R_{ik}^{\alpha}(0) = 0$ is the necessary and sufficient condition that M be without torsion.

The idea of tangent plane further requires that the line elements of T_P issuing from O shall "coincide" with the line elements of M issuing from P; this correspondence must be a one-to-one affine representation. But having already required imbedment of 0th order the method of accomplishing this imbedment of 1st order is fixed. The center O' of $T_{P'}$ arises by the displacement PP' from a definite point O_1 of T_P, and we let the line element $\overline{OO_1}$ on T_P correspond to the line element $\overline{PP'}$ on M. For purposes of calculation it is, however, more convenient to consider the line element $\overline{O_1'O'}$ on $T_{P'}$ which arises from $\overline{OO_1}$ by the displacement PP'. The ξ-coordinates of O_1' on $T_{P'}$ are $d\xi^{\alpha} = -u_i^{\alpha}(0)dx^i$, and consequently

(2) $d\xi^{\alpha} = u_i^{\alpha}(0)dx^i$

are the components of the line element $\overline{O_1'O_1'}$ on $T_{P'}$ or $\overline{OO_1}$ on T_P. The condition that this linear relation between (dx) and $(d\xi)$ be one-to-one reciprocal involves two requirements: (1) the dimensionality m of the tangent plane (which was until now arbitrary) must be the same as the dimensionality n of the manifold M, and (2) the determinant $|u_i^{\alpha}(0)| \neq 0$. If $\mathfrak{G}$ contains the affine group, and we shall henceforth assume that it does, the coordinate system ξ^{α} on T_P can be further adapted to the given coordinate system x^i on M in such a

way that corresponding line elements shall have the same components: $u_i^\alpha(0) = \delta_i^\alpha$.

If $\mathfrak{G}$ were the affine group the previous requirements would fully specify the normal coordinate system ξ^α on T_P in its dependence on the coordinates x^i chosen on M; but this is not the case if $\mathfrak{G}$ is a more extensive group. That is, the "tangent plane" T_P is not as yet uniquely determined by the nature of M, and so long as this is not accomplished we can not say that Cartan's theory deals only with the manifold M. Conversely, the tangent plane in P in the ordinary sense, that is, the linear manifold of line elements in P, is a centered affine space; its group $\mathfrak{G}$ is not a matter of convention. This has always appeared to me to be a deficiency of the theory; I consider that above all, the infinitesimal-geometric researches of Eisenhart, Veblen, T. Y. Thomas, and others in Princeton† have remedied this blemish for projective and conformal geometry.

The connection between ξ and x, although not yet uniquely determined by the previous postulates, allows us to conclude the following: In the development

$$R_{ik}^{\alpha}(\xi) = R_{ik}^{\alpha}(0) + R_{\beta ik}^{\alpha}\xi^\beta + \cdots,$$

the quantities $R_{ik}^{\alpha}(0)$ and $R_{\beta ik}^{a}$ in the point P are determined by the coordinates x^i alone and transform on transformation of coordinates as tensors of order 3 and 4, of the kind indicated by the position of the indices. It is therefore an invariantive restriction to require that our manifold be such that (1) it is without torsion and (2) $\sum_\alpha R_{aik}^{\alpha}$ vanish; we call such a manifold "special."

Let $\mathfrak{G}$ be the projective group. We must then proceed to imbedment of second order in order that T_P be completely

† L. P. Eisenhart, *Non-Riemannian Geometry*, 1927. O. Veblen, *Projective tensors and connections*, Proceedings of the National Academy, vol. 14 (1928), p. 154; *Conformal tensors and connections*, ibid., vol. 14 (1928), p. 735. T. Y. Thomas, *A projective theory of affinely connected manifolds*, Mathematische Zeitschrift, vol. 25 (1926), p. 723.

determined by M. We consider, as an analog, the contact of two surfaces

$$y = f(x^1, \cdots, x^n), \quad y = f'(x^1, \cdots, x^n)$$

in $(n+1)$-dimensional space. Let

$$f' - f = a + a_i x^i + \tfrac{1}{2} a_{ik} x^i x^k + \cdots$$

in the neighborhood of the origin. There is contact of 0th order (intersection) if $a = 0$, of 1st order (tangency) if in addition the linear terms are not present, $a_i = 0$, and finally contact of second order (osculation) if further all a_{ik} vanish. I refer to the two surfaces as semi-osculating if in addition to a and a_i the sum $\sum_{(i)} a_{ii}$, the spur of the quadratic terms, vanishes. Analogously we demand that T_P not only be tangent to the manifold but further that it be semi-osculating. The name tangent plane is then misleading, but we shall use it instead of the more correct "projective semi-osculating plane" for the sake of brevity. The exact definition is as follows. Given an infinitesimal volume element V in P, say a parallelepipedon obtained from line elements in P which shall be infinitely small in comparison with $\overline{PP'} = (dx^i)$. Let V' be the "same" volume element in P', that is, it shall be generated from the line elements with the same components in P'; naturally this construction is dependent on the particular coordinate system employed. Because of the imbedment of order V coincides with a volume element V in O on T_P and V' with one such in O' on $T_{P'}$; this latter is obtained from an element V_1 in O_1 on T_P by the displacement PP'. We now require that V and V_1 have the same volume, measured in the coordinates ξ on T_P.

It is again more convenient for the calculation to take V and V_1 over into V_1', V' on $T_{P'}$ by means of the displacement PP'. The isomorphic representation which carries V' into V_1' is by definition simply (1)—taken for ξ's which are infinitely small compared to the dx^i. Consequently we write

$$u_i^\alpha(\xi) = \delta_i^\alpha + \Gamma_{\beta i}^\alpha \xi^\beta,$$

and on introducing
$$\Gamma^{\alpha}_{\beta i} dx^i = d\gamma^{\alpha}_{\beta},$$
we find
$$\log (V'/V_1') = \log (V_1/V) = \sum_{\rho} d\gamma^{\rho}_{\rho}.$$

Our condition is that this trace shall vanish, and we assert that it can be fulfilled by appropriate choice of the projective coordinates ξ^{α} on T_P. The previous requirements determine the ξ except for a projective transformation of the type

$$\bar{\xi}^i = \frac{\xi^i}{1 + \sum \alpha_k \xi^k},$$

which leaves the center unaltered and is the identity to terms of first order in the neighborhood of the center. The ratio of the measures of volume elements V and $\overline{V}$ in ξ and $\bar{\xi}$, situated at (ξ^{α}), is given by the functional determinant

$$\left| \frac{\partial \bar{\xi}^i}{\partial \xi^j} \right| = (\bar{\xi}\xi).$$

For infinitely small ξ this determinant is

$$(\bar{\xi}\xi) = 1 - (n + 1) \sum \alpha_k \xi^k.$$

Our volume V is at the point $\xi = 0$, V_1 at $\xi^{\alpha} = dx^{\alpha}$, and consequently

$$\log (\overline{V}_1/\overline{V}) = \log (V_1/V) - (n + 1) \sum \alpha_i dx^i$$

from which we see that in order that $\overline{V}_1 = \overline{V}$ the α_i can be chosen in one and only one way; $\alpha_i = \Gamma^{\rho}_{\rho i}$.

The projective coordinate system ξ on T_P is now completely specified by the coordinates x^i on M. If we refer M to a new coordinate system $\bar{x}^i$ we shall have a new projective coordinate system ξ on T_P. The projective transformation $\xi \to \bar{\xi}$ can be described by the facts (1) that it agrees in the neighborhood of the origin with the transformation $x \to \bar{x}$ in terms of first order about P and (2) that the functional determinant $(\bar{\xi}\xi)$ in 0 agrees with $(\bar{x}x)$ in terms of 1st as well as

0th, order. H. P. Robertson* pointed out in a short note that this relation is the decisive point in Veblen's transformation theory of projective space. What we do here, however, is not simply connect the *transformations* of the ξ with those of x but rather we associate a projective ξ *coordinate system* on T_P with an individual x system on M. This possibility arises from the fact that we begin with the projective connection and with its aid tie up the ξ with the x, i.e. accomplish the complete imbedment of the tangent plane. But on the other hand the transformation of the ξ is determined by the transformation of the x, as described above, without taking the given projective connection into account. Veblen's procedure corresponds to this method: this relation between the two transformations is first obtained and the corresponding invariant theory of possible projective connections then developed.

The introduction of $n+1$ homogeneous projective coordinates η by means of the equation $\xi^i = \eta^i/\eta^0$ is for the present purely a matter of convenience. The formulas for the displacement expressed in terms of them have the form

$$(3) \qquad d\eta^\alpha = -\, d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta i}^\alpha dx^i.$$

(From now on Greek indices shall run from 0 to n and Latin from 1 to n.) Since only the ratios of the η are to be considered we can and shall introduce the normalizing condition $d\gamma_0^0 = 0$. We then have $\Gamma_{\rho i}^\rho = \delta_i^\rho$ and $\Gamma_{\rho i}^\rho = 0$. In the case of a special manifold (see above) we have furthermore the symmetry condition $\Gamma_{ki}^\alpha = \Gamma_{ik}^\alpha$ and the Γ with only Latin indices determine the remaining components. This leads to the theorem: *If we allow only special manifolds the projective connection is determined uniquely by the geodesics.*†

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* H. P. Robertson, *Note on projective coordinates*, Proceedings of the National Academy, vol. 14 (1928), p. 153.

† See J. A. Schouten, Rendiconti di Palermo, vol. 50 (1926), pp. 142–169, in particular p. 158. I do not find that this work, which is closely related to our process, gives a clear account of the fact that the coordinates ξ must be tied up with the x as described above.

For the complete development of projective infinitesimal geometry we must, in my estimation, add three independent ideas to Cartan's scheme; the first and most important of these consists in connecting the coordinates ξ with the x by the requirement of "semi-osculation," the second answers the question to what extent the geodesics determine the projective connection by the invariantive "specialization." The third idea, which I now consider, is due to T. Y. Thomas: it is possible to give the variables η themselves, and not only their ratios, a geometrical interpretation. The analytic expression in coordinates η for any projective mapping which leaves O invariant,

$$(4) \qquad \bar{\eta}^i = \sum_k a_k{}^i \eta^k, \quad \bar{\eta}^0 = \eta^0 + \sum_k \alpha_k \eta^k,$$

can be so normalized that the coefficient $a_0^0 = 1$. This normalization is useful because it is not destroyed by composition: the group $\mathfrak{G}_0$ is replaced by the isomorphic group of affine transformations in $n+1$ dimensions of the form (4). If the transition $\eta \to \bar{\eta}$ on T_P corresponds to the transition $x \to \bar{x}$ on M the transformation (4) can further be described as agreeing with the transformation of the differentials $dx^0, dx^1, \cdots, dx^n$ in P when the additional coordinate x^0 transforms in accordance with the law

$$(5) \qquad \bar{x}^0 = x^0 - \frac{1}{n+1} \log (\bar{x}x).$$

We now have an $(n+1)$-dimensional manifold M^* instead of M; each point of M is replaced by a filament of M^* along which $x^1, \cdots, x^n$ are constant and only x^0 varies. By (5) the distance between points on the same filament, i.e. the difference of their x^0 coordinates, as well as the filaments themselves, have an invariantive significance. An $(n+1)$-dimensional affine tangent plane, the domain of the variables η, is associated with each point of M^*. The transformation of the η, which is related to a transformation of the x on M, is the same for all points on the same filament. Extending the Γ by adding $\Gamma_{\beta 0}^\alpha = \delta_\beta^\alpha$, this means that the ratios $\eta^0 : \eta^1 : \cdots : \eta^n$

of a point on the tangent plane are unaltered by displacement along the filament. The n-dimensional projective displacement on M^* defined by

(6) $$d\eta^\alpha = -\ d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta\rho}^{\ \ \overset{\rho}{\ }} dx^\xi,$$

is consequently invariantively determined by the projective displacement on M.

We must next ask if this is also true for the $(n+1)$-dimensional affine displacement expressed by the same formulas; the answer is affirmative, because our normalization is so chosen that $\Gamma_{\beta\rho}^{\ \alpha}$ is symmetric in β and ρ. To show this, let $\overline{\Gamma}_{\beta\rho}^{\ \alpha}$ be the projective connection of M evaluated in a new coordinate system $\bar{x}^i$ in the manner described above, and let $\tilde{\Gamma}_{\beta\rho}^{\ \alpha}$ be the components of the same affine connection on M^* expressed in terms of the coordinates $\bar{x}^i$ in the manner indicated by (6) in terms of the x^i. Then the corresponding equations (6) characterize the same projective connection, that is, $\overline{\Gamma}_{\beta\rho}^{\ \alpha}$ and $\tilde{\Gamma}_{\beta\rho}^{\ \alpha}$ can differ only by a term of the form $\delta_\rho^\alpha \lambda_\rho$. Now $\overline{\Gamma}$ as well as $\tilde{\Gamma}$ must be symmetric in the two lower indices and consequently

$$\delta_\beta^\alpha \lambda_\rho = \delta_\rho^{\ \alpha} \lambda_\beta,$$

from which we obtain by the contraction $\alpha = \rho$

$$\lambda_\beta = (n+1)\lambda_\beta, \quad \lambda_\beta = 0.$$

All that we have said here can be taken over mutatis mutandis to the conformal geometry. Here the equation

(7) $$g_{ik}dx^i dx^k = 0$$

on M is fundamental. If g_{ik} be the values of the coefficients in P then the conformal geometry on the homogeneous "tangent" plane T_P is described by the equation $g_{ik}d\xi^i d\xi^k = 0$ with constant coefficients g_{ik}; the group $\mathfrak{G}$ consists of all transformations of the ξ which leave this equation invariant and consequently depends on the point P in question. The conformal displacement PP' must be a transformation which takes the equation $g_{ik}d\xi^i d\xi^k = 0$ over into $g_{ik}'d\xi'^i d\xi'^k = 0$. Consequently this does not agree literally with the scheme de-

veloped above—furthermore $\mathfrak{G}$ does not contain the affine group (only the orthogonal one). We must expressly require that the coincidence of the line elements $\overline{PP'}$ on M and $\overline{OO_1'}$ on T_P is a conformal, and not merely an affine, relation. (1) The requirement of semi-osculation is also here sufficient to tie up the conformal coordinate system ξ uniquely with the x. (2) If the manifold is special the conformal connection is uniquely determined by equation (7). (3) The transition to homogeneous coordinates η, in which the conformal representation appears as a homogeneous linear transformation, is here accomplished, following Möbius, by

$$\eta^0:\eta^1: \cdots :\eta^n:\eta = 1:\xi^1: \cdots :\xi^n: -\tfrac{1}{2}g_{ik}\xi^i\xi^k.$$

These coordinates are subject to the relation

$$g_{ik}\eta^i\eta^k + 2\eta^0\eta = 0.$$

It is convenient for the purpose of calculation to normalize the coefficients g_{ik}, only the ratios of which are given, by the condition $|g_{ik}| = 1$. This third and last step, which was carried through by Veblen in a recent paper, proceeds as before, but the result is more complicated since we have $n+2$ variables η, whereas Thomas' extension gives but $n+1$ coordinates x. Consequently we do not arrive at an affine connection on an $(n+1)$- or $(n+2)$-dimensional manifold M^*, invariantly related to the conformal connection on M.

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