ON CERTAIN LOCI OF LINES INCIDENT WITH CURVES AND SURFACES IN FOUR-SPACE

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In this paper is presented a partial list of formulas giving the orders of the ruled hypersurfaces and ruled surfaces whose rulings are incident with given curves and surfaces in 4-space. If the number of incidences is equivalent to six simple conditions, the number of lines having these incidences is finite. The formulas herein presented are analogous to those giving the orders of the surfaces whose rulings are lines satisfying three conditions and the number of lines satisfying four conditions in 3-space.* Some of these formulas are obvious, others require proof. Those that are not herein included are left out for future consideration.

We shall let h be the number of apparent double points, p the deficiency, of a given curve C^m; H the number of apparent double points, P the deficiency, of a 3-space section c^n of a given surface F^n; and t the number of apparent triple points of F^n or the number of lines through a given point meeting F^n three times.†

First consider the hypersurface V^μ whose lines satisfy four conditions. For a line to meet a given curve once is equivalent to two conditions. Hence the ∞^2 lines cutting across two given curves C^{m_1}, C^{m_2} form a hypersurface whose order is obviously

$$(1) \qquad \mu = m_1 m_2 .$$

* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. II, §458–475.

† The formula for t is $\mu\nu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/6$ if F^n is the complete intersection of two hypersurfaces of orders μ and ν respectively. If F^n is only a partial intersection, it is necessary to find t by some other means. See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35 (1929), pp. 339–343.

If the two curves have s points in common, s must be deducted from the above formula. The order, then, is equal to the number of apparent intersections of the two curves.

The order of the bisecant hypersurface of a given C^m is equal to the number of apparent double points of C^m, that is,

$$(2) \qquad \mu = h = \frac{(m-1)(m-2)}{2} - p \cdot$$

The multiplicity of C^m on the hypersurface is, as we shall see, $m-2$.

The V^μ whose ∞^2 lines meet a curve C^m and two surfaces F^{n_1}, F^{n_2} each once is of order

$$(3) \qquad \mu = 2mn_1n_2.$$

The proof of this formula is analogous to that of the formula for the order of the surface whose rulings meet three given curves in S_3.*

If C^m meets F^{n_1} and F^{n_2} in s_1 and s_2 points respectively and if F^{n_1} and F^{n_2} meet in a curve of order k, then the quantity $s_1n_2+s_2n_1+km$ has to be deducted from the formula.

The bisecants of an F^n that meet a C^m form a locus of order

$$(4) \qquad \mu = m[H + n(n-1)/2] = m[(n-1)^2 - P].$$

The quantity $s(n-1)$ has to be deducted if C^m and F^n have s points in common.

The ∞^2 lines that are incident with four given surfaces of orders n_1, n_2, n_3, n_4 respectively form a V^μ of order

$$(5) \qquad \mu = 3n_1n_2n_3n_4\dagger.$$

Each of the four surfaces is of multiplicity equal to the product of the orders of the other three and each of the points common to two of the surfaces is of multiplicity equal to the sum of their orders multiplied by the product of the orders of the other two surfaces.

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* Salmon, loc. cit., §467.

† For the case $n_1=n_2=n_3=n_4=1$, see Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36.

The order of the V^μ whose lines meet a given surface F^n twice and two given surfaces F^{n_1}, F^{n_2} each once is

$$(6)\quad \mu = n_1n_2[2H + n(n-1)/2] = n_1n_2[(n-1)(3n-4) - 4P]/2.$$

To prove this formula, obtain the order μ' of the hypersurface whose lines are incident with two planes π, π' and with F^n twice. An S_3 through π' meets π in a line l and F^n in a curve c^n. The lines that meet l once and c^n twice form a surface of order $H+n(n-1)/2$. Since l is H-fold on the surface, π is H-fold on $V^{\mu'}$. For the same reason π' is H-fold. Since S_3 meets $V^{\mu'}$ in a composite surface made up of π' counted H-ply and a surface of order $H+n(n-1)/2$, $V^{\mu'}$ is of order

$$\mu' = 2H + n(n-1)/2.$$

Replacing π, π' by F^{n_1}, F^{n_2}, we obtain the formula above.

To obtain the order of the V^μ whose lines are trisecants of an F^n and are incident with an F^{n_1}, we note that, t being the number of apparent triple points on F^n, F^{n_1} is t-fold on the hypersurface. Since $(n-2)(6H+n-n^2)/6$ is the order of the trisecant surface* of an S_3-section of F^n, we have the formula

$$(7)\quad \mu = n_1[t + (n-2)(6H + n - n^2)/6].$$

Now we pass to the scrolls whose rulings are lines satisfying five simple conditions. The order of the scroll of lines meeting a curve C^m once and three surfaces F^{n_1}, F^{n_2}, F^{n_3} each also once is

$$(8)\quad \nu = 3mn_1n_2n_3.$$

To see this, consider the surface $F^{\nu'}$ whose lines meet a given line l and the three surfaces F^{n_1}, F^{n_2}, F^{n_3}. Any S_3 through l meets F^{n_1}, F^{n_2}, F^{n_3} in three curves c^{n_1}, c^{n_2}, c^{n_3} respectively. The order of the surface in S_3 whose lines meet c^{n_1}, c^{n_2}, c^{n_3} is $2n_1n_2n_3$. Since from each point of l, $n_1n_2n_3$ lines can be drawn incident with F^{n_1}, F^{n_2}, F^{n_3}, l is to be regarded as $n_1n_2n_3$-fold

* Salmon, loc. cit., §471.

on $F^{\nu'}$. Hence $F^{\nu'}$ is of order $\nu'=3n_1n_2n_3$, for it is met by S_3 in the line l counted $n_1n_2n_3$-ply and the $2n_1n_2n_3$ lines that cut across l, c^{n_1}, c^{n_2}, c^{n_3}. If we replace l by C^m, we have formula (8).

The following formulas can be proved in a similar manner:

$$\nu = mn_1[2H + n(n-1)/2] \tag{9}$$

for the order of the scroll whose lines meet C^m once, F^n twice, and F^{n_1} once;

$$\nu = m[t + (n-2)(6H + n - n^2)/6] \tag{10}$$

for the order of the surface whose lines are trisecants of F^n and unisecants of C^m;

$$\nu = n[h + m(m-1)/2] = n[(m-1)^2 - p] \tag{11}$$

for the order of the surface formed by the bisecants of C^m that are incident with F^n; and

$$\nu = 2m_1m_2n \tag{12}$$

for the order of the surface whose rulings meet C^{m_1}, C^{m_2}, F^n each once.

The orders of the surfaces whose generators meet five surfaces of orders n_1, n_2, n_3, n_4, n_5 each once; one surface F^n twice and three surfaces of orders n_1, n_2, n_3 each once; and one surface F^n three times and two surfaces of orders n_1, n_2 each once are, respectively,

$$\nu = 5n_1n_2n_3n_4n_5, \tag{13}$$

$$\nu = n_1n_2n_3(3H + n^2 - n), \tag{14}$$

$$\nu = n_1n_2[t + (n-2)(6H + n - n^2)/3]. \tag{15}$$

Passing to the formulas for the number of lines that satisfy six simple conditions, we first consider the number N of trisecants to a curve C^m. Let C^m be made up of m lines l_i such that each meets its consecutive line, l_1 and l_m being skew. The number of triples of lines that are all skew is equal to the number of lines which meet a proper C^m, assumed rational, three times. This number is $(m-2)(m-3)(m-4)/6$. If C^m is of deficiency p, the quantity $(m-4)p$ is to be deducted. Hence the formula for the number of trisecants to a curve of order m and deficiency p is

$$(16) \qquad N = \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p \cdot$$

The number of lines that meet C^m twice and C^{m_1} once is, from (2),

$$(17) \qquad N = m_1 h = m_1\left[\frac{(m-1)(m-2)}{2} - p\right] \cdot$$

We now prove that the multiplicity of C^m on its bisecant hypersurface V^h [see formula (2)] is $m-2$. A line l incident with C^m meets V^h in $h-x$ points distinct from C^m, where x is the multiplicity of C^m. There are $h-x$ lines that meet C^m twice and l once. This number plus the number, $(m-2) \cdot (m-3)(m-4)/6-(m-4)p$, of lines that meet C^m three times is equal to the number of lines that meet a curve C^{m+1} of order $m+1$ and deficiency p three times. Hence

$$h - x + \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p$$
$$= \frac{(m-1)(m-2)(m+3)}{6} - (m-3)p,$$

the right-hand member being the result of changing m to $m+1$ in (16). Replacing h by $[(m-1)(m-2)/2]-p$, we have $x=m-2$, which was to be proved.

The following formulas require no explanation. From (1), (3), (4) we obtain, respectively,

$$(18) \qquad N = m_1 m_2 m_3$$

for the number of lines that meet three curves of orders m_1, m_2, m_3, each once;

$$(19) \qquad N = 2m_1 m_2 n_1 n_2$$

for the number of lines incident with two curves of orders m_1, m_2 and two surfaces of orders n_1, n_2;

$$(20) \qquad N = mm_1[(n-1)^2 - P]$$

for the number of lines that meet two curves of orders m, m_1 each once and one surface of order n twice. Formulas (5) or (8), (6) or (9), and (7) or (10) yield, respectively,

$$N = 3mn_1n_2n_3n_4 \tag{21}$$

lines meeting a curve of order m once and four surfaces of orders n_1, n_2, n_3, n_4 each once;

$$N = mn_1n_2[2H + n(n-1)/2] \tag{22}$$

lines meeting C^m, F^{n_1}, F^{n_2} each once and F^n twice; and

$$N = mn_1[t + (n-2)(6H + n - n^2)/6] \tag{23}$$

lines that meet F^n three times, F^{n_1} and C^m each once.

We also have the formulas

$$N = nn_1[(m-1)^2 - p] \tag{24}$$

for the number of bisecants of C^m that meet F^n, F^{n_1} each once;

$$N = 5n_1n_2n_3n_4n_5n_6 \tag{25}$$

for the number of lines incident with six surfaces;

$$N = n_1n_2n_3n_4(3H + n^2 - n) \tag{26}$$

for the number of lines meeting F^n twice and four other surfaces each once; and

$$N = n_1n_2n_3[t + (n-2)(6H + n - n^2)/3] \tag{27}$$

for the number of trisecants of F^n that are incident with three other surfaces.

There are other formulas that call for attention. We omit them for the present.

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