THE MARCH MEETING IN NEW YORK

The two hundred sixty-eighth regular meeting of the American Mathematical Society was held at Columbia University, on Friday and Saturday, March 29–30, 1929, extending through the usual morning and afternoon sessions. The attendance included the following one hundred thirty-two members of the Society:

C. R. Adams, A. A. Albert, R. L. Anderson, R. G. Archibald, H. E. Arnold, Bamforth, Bellamy, A. A. Bennett, Benton, A. C. Berry, William James Berry, William Johnston Berry, Birkhoff, Bowden, B. H. Camp, G. A. Campbell, Carlitz, Cope, L. P. Copeland, Cramlet, Dehn, Demos, Doak, Doermann, Dresden, Eisenhart, Emch, Engstrom, Feld, Fite, D. A. Flanders, Fort, R. M. Foster, Philip Franklin, Frink, Gale, Gehman, Gergen, Gill, Gronwall, C. C. Grove, G. H. Hardy, Hazeltine, Hebbert, Hildebrandt, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Huber, R. L. Jackson, R. A. Johnson, Kaplan, Kasner, O. D. Kellogg, Kenny, Kholodovsky, Rose Klein, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Kunte, Lamson, Langman, Lefschetz, Lehr, Littauer, Litzinger, Lubin, MacColl, MacInnes, Marden, Meder, Michal, E. J. Miles, H. H. Mitchell, Molina, C. N. Moore, T. W. Moore, Richard Morris, Mullins, F. H. Murray, C. A. Nelson, Newman, Ore, F. W. Owens, H. B. Owens, Parsons, Pell-Wheeler, Pfeiffer, Pooler, Post, R. G. Putnam, Raudenbush, Reddick, M. S. Rees, Rider, Ritt, Robertson, Seely, Serghiesco, Sheffer, Simons, Singer, C. E. Smith, P. A. Smith, S. E. Smith, M. H. Stone, Swartzel, Tamarkin, Evan Thomas, J. M. Thomas, J. E. Thompson, Trjitzinsky, Veblen, Wahlert, H. E. Webb, D. W. Weeks, Weida, Weinstein, Weisner, Weyl, Widder, Wiener, W. J. Willis, J. W. Young, Mabel M. Young, Margaret M. Young, Zippin.

There was no meeting of the Council or of the Board of Trustees. A telegram of greeting was received from the meeting of the Society held at the same time in Chicago, and was acknowledged appropriately.

The afternoon sessions on Friday and Saturday were devoted to a Symposium on Fourier Series. The following papers were presented: Friday, March 29, *Modern work in the theory of ordinary trigonometric series*, by Professor G. H. Hardy, with discussion by Professors Einar Hille and C. N. Moore, and Dr. T. H. Gronwall; Saturday, March 30, *Fourier series and almost periodic functions from the standpoint* *of the theory of groups*, by Professor Hermann Weyl, with discussion by Professors Norbert Wiener, T. H. Hildebrandt, Philip Franklin, and M. H. Stone. Ex-President Birkhoff presided at the Symposium, relieved during the discussion on Saturday afternoon by Professor Kasner.

The morning sessions were devoted to the reading of short papers. Vice-President Young presided on Friday morning, relieved by Professor Ritt, and Professor H. H. Mitchell on Saturday morning. Titles and abstracts of the papers read at these sessions follow below. The papers of Bamforth (except his joint paper with Birkhoff), Browne, Cope (second paper), Féraud, Gronwall (second paper), Hickey, Lubin, Michal (second paper and joint paper with Peterson), Peterson, Pierpont, Rutledge, Seely, Vandiver, Walsh, Wiener were read by title. Dr. Féraud was introduced by Professor Birkhoff, Miss Hickey by Professor Michal, Mr. Hurwitz by Professor Ritt, Mr. Thielman by Professor Michal, and Mr. Whitney by Professor Pierpont.

1. Professor C. R. Adams: *Note on integro-q-difference equations.*

The equation considered is one in which the unknown function $f(x)$ is subjected simultaneously to a q-difference operator of the nth order and to an integral operator of Volterra type. Certain facts about the solutions of the equation of first order when $|q| \leqq 1$ follow from the work of Picone, Popovici, Nalli and Tamarkin; these writers not unnaturally confined themselves to the real domain. The main purpose of this paper is to investigate the existence of solutions in the complex domain for the equation of order n including the case of $|q|<1$ and the essentially different case of $|q|>1$. Assuming the known functions in the equation to be analytic we find formal series solutions. When $|q| >1$ the existence of analytic solutions is shown by direct proof of convergence. In the case of $|q|<1$ results are obtained both by a generalization of Tamarkin's work and by a method of successive approximation; these methods also are employed in the real domain when the known functions are subject to less stringent hypotheses.

2. Dr. I. M. Sheffer (National Research Fellow): *The linear functional operators and equations associated with sets of polynomials.*

The writer has considered in a previous report certain algebraic properties of sets of polynomials $P: \{P_n(x)\}$, $n=0, 1, \cdots$, where the degree of $P_n(x)$ does not exceed n. Now we consider certain linear functional operators and equations which arise when we turn to the analytic aspect of such sets. An operator L is defined as carrying the identity set I into P. We can express L as a linear differential operator of infinite order. With L we consider a second operator $\mathcal{L}$, and we investigate the characteristic numbers and characteristic functions of these operators, the problem of expanding "arbitrary" analytic functions in terms of these characteristic functions, and the question of solving the functional equations which are determined by the operators.

3. Professor C. N. Moore: *On the absolute convergence of the double Fourier series.*

In this paper it is shown that if a function of two variables which is of bounded variation also satisfies at a point a condition which is a generalization of the Lipschitz condition for functions of one variable, the double Fourier series corresponding to this function will be absolutely convergent at the same point. This theorem is a generalization to double Fourier series of a criterion found by Zygmund (Journal of the London Mathematical Society, vol. 3 (1928), p. 194) for the ordinary Fourier series.

4. Dr. F. R. Bamforth (National Research Fellow) and and Professor G. D. Birkhoff: *Divergent series and singular points of ordinary differential equations.*

Consider the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ in which the X_i are analytic in x_i for $|x_i| \leqq r$, $r>0$ and $X_i(0, 0, 0)=0$, $i=1, 2, 3$. Let the equations of variation be $dy_i/dt = \sum_{j=1}^{3} c_{ij}y_j$ where the constants c_{ij} are the values of $\partial X_i/\partial x_j$ evaluated at $x_1 = x_2 = x_3 = 0$. Let the roots m_1, m_2, and m_3 of the equation $|c_{ij} - m\delta_{ij}| = 0$ be such that m_1 and m_2 are greater than zero, m_3 is less than zero, and there exists among them no relationship of the form $m_i = p_1m_1 + p_2m_2 + p_3m_3$, where the p_i are integers, positive or zero, whose sum is greater than 1. In this paper it is shown that under these hypotheses, in a certain neighborhood of the origin $x_1 = x_2 = x_3 = 0$, the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ is equivalent to the system $dz_i/dt = m_iz_i$, $i=1, 2, 3$, by means of a transformation of the type $z_i = z_i(x_1, x_2, x_3)$ where the functions $z_i(x_1, x_2, x_3)$ have the properties that they are zero for $x_1 = x_2 = x_3 = 0$, are of class C^∞, and are analytic at every point (x_1, x_2, x_3) which does not lie on a certain set of three surfaces passing through the origin.

5. Dr. F. R. Bamforth: *A boundary-value problem for a system of ordinary differential equations not linear in the parameter.*

In matrix notation in which small letters will stand for vectors of order n, capital letters for square matrices of order n, and Greek letters for numbers, the boundary-value problem studied may be written in the form $y' = y(A + B_1/(\lambda - \beta_1) + B_2/(\lambda - \beta_2))$, $My(a) + Ny(b) = 0$, where $\beta_1 \neq \beta_2$. The elements of the matrices A, B_1, B_2 are real single-valued, continuous functions of x on the interval $a \leqq x \leqq b$, λ is a parameter, and the elements of the matrices M and N as well as β_1 and β_2 are real numbers. For boundary-value problems of the above form, definitions of adjointness and of definite self-adjointness are given. For definitely self-adjoint boundary-value problems questions relative to the existence of characteristic parameter values and the possibility of expanding an arbitrary vector in terms of the characteristic solutions are considered.

6. Dr. F. R. Bamforth: *A boundary-value problem for a system of differential equations of the second order.*

In a paper presented by the author at the Chicago meeting of the society last spring, a set of normal forms was derived for boundary-value problems connected with systems of ordinary differential equations which are linear in a parameter λ. This paper brought to light certain boundary-value problems of general type which had never been studied. The present paper is a study of one of these problems, which may be written in the form $y_1' = (a_{11}(x)+\lambda)y_1+a_{12}(x)y_2$, $y_2' = \lambda(x)y_1-(a_{11}(x)-\lambda)y_2$, $m_{11}y_1(a)+m_{12}y_2(a)+n_{11}y_1(b)+n_{12}y_2(b)=0$ $m_{21}y_1(a)+m_{22}y_2(a)+n_{21}y_1(b)+n_{22}y_2(b)=0$. Under certain hypotheses on the coefficients, it is found that there exists a denumerable infinitude of characteristic numbers. Furthermore, sufficient conditions for the simultaneous expansion of a pair of fairly arbitrary functions in terms of the characteristic solutions are given.

7. Mr. B. P. Gill: *An analog for algebraic functions of the Thue-Siegel theorem.*

In this paper the following theorem is proved. Let $\theta(t)$ be an algebraic function of t defined by an irreducible equation of degree $n \geqq 2$. Then there exist only a finite number of rational functions $x(t)/y(t)$, (x and y polynomials in t), such that, if the difference $\theta(t)-x(t)/y(t)$ be expanded about infinity in a series of descending powers of t, the initial term of the series is of degree less than $-2n^{1/2}\times$the degree of y. Applications are indicated to indeterminate algebraic equations in which the unknowns represent polynomials.

8. Mr. Solomon Hurwitz: *Algebraic first integrals of algebraic differential equations.*

It was shown by Fuchs that if an algebraic differential equation $f(x, y, y')=0$, or $y'=u(x, y)$, has a general solution $V(x, y)=c$, where V is an algebraic function of x and y, then there exists a general solution $R(x, y, u)=c$, R being a rational function of x, y, and u. In the present paper, it is shown that if an algebraic differential equation of the nth order $f(y^{(n)}, y^{(n-1)}, \cdots, y', y, x)=0$, or $y^{(n)}=u(y^{(n-1)}, \cdots, y', y, x)$, has an algebraic first integral $V(y^{(n-1)}, \cdots, y', y, x)=c$, then there exists a first integral $R(y^{(n-1)}, \cdots, y', y, x, u)=c$, where R is a rational function of the quantities in parentheses.

9. Dr. T. F. Cope (National Research Fellow): *Sufficient conditions for a weak minimum for a special case of the problem of Mayer with variable end points.*

The problem considered is that of finding conditions which suffice for a curve $y_i=y_i(x)$, $x_1 \leqq x \leqq x_2$, $i=1, \cdots, n$, to minimize the first of a set of functions $f_\rho[x_1, x_2, y_j(x_1), y_j(x_2)]$, $\rho=1, \cdots, r \leqq 2n+2; j=1, \cdots, n$, in the class of similar curves which make $f_2, \cdots, f_r$ vanish and besides satisfy the differential equations $\theta_\alpha(x, y_j, y_j')=0$, $\alpha=1, \cdots, m<n$. First necessary conditions, namely the Euler equations and transversality conditions, were deduced by Bliss (Transactions of this Society, vol. 19 (1918), p. 305). In the author's Chicago dissertation, presented to this Society last year at the April meeting in Chicago, another necessary condition was found, which is essentially that the characteristic numbers of a boundary-value problem associated with the second variation must be greater than or equal to zero. If x_1 and x_2 are fixed, it is shown in the present paper that sufficient conditions for a weak relative minimum can be deduced from the necessary conditions mentioned above, the comparison curves being of class C'.

10. Dr. T. F. Cope: *A set of sufficient conditions for a minimum for the problem of Mayer with variable end points for specialized comparison arcs.*

The problem considered is that of the preceding paper. If the comparison curves are defined suitably, it is proved that the necessary conditions mentioned in the first paper give rise to conditions which are sufficient for a weak relative minimum even when x_1 and x_2 as well as $y_i(x_1)$, and $y_i(x_2)$ are permitted to vary. The boundary-value problem associated with the second variation in this case is of an essentially more complicated type than that of the first paper, and special methods are required to discuss it.

11. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.*

The conditions that in an Einstein parallelism geometry the straight lines and the geodesics shall coincide are set up, and the resulting partial differential equations for the Z's completely integrated in the case of three dimensions.

12. Professor D. V. Widder: *On the composition of the singularities of functions defined by factorial series.*

The purpose of the present paper is to determine the form of the coefficients c_n in the series $F(s)=\sum c_n n!/[s(s+1)\cdots(s+n)]$ so that the function $F(s)$ which is defined by the series shall have singularities at most at the points $\alpha+\beta$, where the points α are the singularities of the function $f(s)=\sum a_n n!/[s(s+1)\cdots(s+n)]$, the points β those of the function

$\phi(s)=\sum b_n n!/[s(s+1)\cdots(s+n)]$. It is found under certain specified conditons that c_n may be taken as $a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$, the nth coefficient of the power series expansion of the product of the two power series $\sum a_ns^n$ and $\sum b_ns^n$. The result is obtained by expressing $f(s)$ and $\phi(s)$ as generating functions of the form $\int_0^\infty e^{-st}\alpha(t)dt$, and applying a result previously obtained and presented to the Society by the author.

13. Dr. W. J. Trjitzinsky: *On composition of singularities.*

Several theorems on composition of singularities of the type analogous to Hadamard's multiplication theorem are derived. The method is based essentially on complex integration, and on the possibility of deforming a contour unless singularities of a certain integrand tend to coincide. Let α and β be the respective singularities of two functions. Among the theorems obtained are two about a third function, in the first theorem with singularities of the form α with the exponent β, and in the second theorem with singularities of the form $(a\alpha\beta+b\alpha+c\beta+d)/(e\beta+f)$. An extension of the latter theorem, which is a generalization of Hadamard's theorem, is given for functions of two variables, similar to Haslam-Jones' extension of the ordinary multiplication theorem

14. Professor O. D. Kellogg: *Singular manifolds among those of an analytic family.*

This paper will appear in full in an early issue of this Bulletin.

15. Dr. C. M. Cramlet (National Research Fellow): *Invariants of the n-ary q-ic differential form.*

The general differential form may be written as $a_{\rho_1\cdots\rho_q}dx^{\rho_1}\cdots dx^{\rho_q}$, where the $a_{r_1\cdots r_q}$ are functions of the n variables $x^1,\cdots,x^n$. Subjecting the x's to an analytic transformation and assuming the form to be invariant implies a sequence of tensors. The problem of obtaining these tensors is reduced to a problem of an affine connection Γ^a_{bc} and a tensor $a_{r_1\cdots r_q}$, where the Γ^a_{bc} are functions of position and are given explicitly in terms of the $a_{r_1\cdots r_q}$ and their derivatives. Covariant differentiation may then be defined as in affine geometry and a complete set of tensors are the $a_{r_1\cdots r_q}$ and their successsive covariant derivatives, and the curvature tensor B^r_{stu} and its covariant derivatives. The equivalence of two forms is stated in terms of a finite number of these tensors. The condition under which the present solution can be effected is that the discriminant of the form is non-vanishing. It is expected that this condition can be replaced by the condition that there exists at least one non-vanishing algebraic invariant of the tensor $a_{r_1\cdots r_q}$. Some interpretations of the significance of the tensors are given.

16. Professor A. D. Michal: *Projective functional invariants.* Preliminary communication.

This paper is concerned with the construction of functionals that are the function-space analogs of the projective invariants of n-dimensional affinely connected manifolds. A generalization of Weyl's projective curvature tensor is obtained.

17. Professor A. D. Michal: *Tensor analysis in function space under functional transformations with differentials of the third kind.*

In an earlier paper (American Journal of Mathematics, vol. 50 (1928), pp. 473–517) the author has shown how to construct a theory of tensors in function space. In the present paper this theory is generalized and made a more harmonious whole by the employment of functional transformations with differentials of the third kind, that is, (1): $\delta z^i = {}_i{}'z^{(i)}\delta\bar{z}^i + {}'z_\alpha{}^i\delta\bar{z}^\alpha$. The differential functional forms are taken in the general normal form with generalizations in the differentials of their coefficients appropriate to (1).

18. Professor A. D. Michal and Mr. T. S. Peterson: *The invariant theory of functional forms under the group of linear functional transformations of the third kind.*

The invariant theory of special functional forms under the Fredholm group was initiated by A. D. Michal (American Journal of Mathematics, vol. 50 (1928)). Further contributions to this subject were given by T. S. Peterson (this Bulletin, vol. 34 (1928), p. 416). This paper is concerned with the development of an invariant theory of still more general "algebraic" functional forms in addition to the more general underlying group, that of linear functional transformations of the third kind.

19. Mr. H. P. Thielman: *On new integral addition theorems for Bessel functions and series of the hypergeometric type.*

Although Bessel functions and hypergeometric series have been studied in great detail, the literature seems to have very little to say about their integral addition theorems. This paper derives a large number of such addition theorems by employing Volterra's theories of functions of composition.

20. Miss Maude I. Hickey: *Representations of groups of linear functional transformations.*

This paper begins the systematic study of representations of groups of linear functional transformations in one function by means of linear functional transformations in a sequence of functions of several variables.

21. Mr. T. S. Peterson: *Note on the law of transformation of the nth resolvent.*

This note supplements in full the law of transformation of the nth resolvent as defined in an earlier paper (see T. S. Peterson, this Bulletin, vol. 34 (1928), p. 416).

22. Professor James Pierpont: *On the attraction of spheres in elliptic space.*

This paper has appeared in full in the May-June number of this Bulletin.

23. Professor George Rutledge: *On the relation between de la Vallée-Poussin summability and the cardinal series of interpolation.*

W. L. Ferrar (Proceedings of the Royal Society of Edinburgh, vol. 45 (1925), p. 269) has proved that, under certain very general conditions, the cardinal series of interpolation is summable by the method of de la Vallée-Poussin to the sum of the Gauss series of even order, whenever the latter series is convergent. Restricting conditions are inherent in Ferrar's method of proof, but do not in fact exist. The Gauss series of even order is identically equal, at any stage of summation, to the Lagrange polynomial of even degree, and when the latter is written in the form (previously presented by the author) $x\prod_{k=1}^{n}(1-x^2/k^2)\{f(0)/x+\sum_{r=1}^{n}\binom{2n}{n-r}\binom{2n}{n}^{-1}(-1)^r \cdot [f(r)/(x-r)+f(-r)/(x+r)]\}$, it becomes at once apparent that the sum within the braces is nothing other than the sum of $n+1$ terms of the sum (VP) of the cardinal series. Thus summability (VP) of the cardinal series and convergence of the Lagrange polynomial of even degree (and hence of the series of Gauss and Stirling, of even order) involve exactly the same limit and are equivalent without restriction.

24. Professor P. R. Rider: *On the distribution of the ratio of mean to standard deviation in small samples.*

The ratio of the mean of a sample (measured from the mean of the universe) to the standard deviation of the sample plays an important part in a number of statistical problems such as determining the probability that the mean of the sample does not deviate from the mean of the universe by more than a specified amount, comparing two mean values, and finding the sampling errors of regression coefficients. The distribution of this ratio for the case of a normal universe has been completely derived. The derivation, however, depends upon certain assumptions which do not hold for other than normal universes. The present paper derives the distribution of the ratio for certain discrete universes, with particular reference to a rectangular universe. Incidentally the distributions of other statistical parameters (mean, median, extreme average, least variate, range, and variance) for small samples are discussed.

25. Professor B. H. Camp: *Some corollaries of Bayes' theorem.*

If the incomplete Beta function is inserted as the unknown frequency function in Bayes' theorem, two results follow immediately. One is that, if m successes have occurred in n trials, the probability of a success on the next trial is quite arbitrary, as it should be, dependent on the character of the universe from which the n trials were selected. This result compels attention to the indeterminateness of the theorem. What is frequently only a tacit assumption is now so introduced as to stand out baldly in the concluding formula. Another result is that, if, as sometimes happens, one actually does know something about the original universe from which the material under investigation was selected, this knowledge can be utilized. A further discussion is made to answer the question as to precisely what assumptions with regard to the universe the actuary and the physicist are tacitly making when they assume that the true values are near those found by experiment.

26. Professor Oystein Ore: *Remarks on the arithmetic theory of abelian fields.*

This paper contains an application of the author's arithmetic theory of Galois fields (Mathematische Annalen, vol. 100) to abelian fields. The complete system of prime-ideal types is determined, and a new proof is given for the theorem of Kronecker that all abelian fields are cyclotomic fields.

27. Dr. A. A. Albert (National Research Fellow): *The rank function of any simple algebra.*

The rank function of the general simple algebra A over F is considered. We define the central field H of A to be the field of all elements of A commutative with every element of A, and represent A as a normal algebra over H. Then the order of A over H is $q=t^2r^2$, and the rank function of A is the rank function of an associated total matric algebra A' over an extended field H', has degree tr, is irreducible in the function field $K=H(\xi_1, \xi_2, \cdots, \xi_q)$ of its coefficients, and has the symmetric group with respect to K. Then, if the order of H with respect to F is s, the order of A over F is st^2r^2, and the rank function of A in F has degree str and is irreducible in the function field of its coefficients.

28. Dr. A. A. Albert: *The structure of algebras which are the direct products of rational generalized quaternion division algebras.*

Attempts were made, before Cecioni constructed his normal division algebras, to construct new division algebras in sixteen units by considering algebras which were the direct products of any two rational generalized quaternion division algebras. Such attempts are here shown to have been futile, as such a direct product is never a division algebra. The resulting structure of these direct products is also considered and the theorems generalized to direct products of any number of rational generalized quaternion division algebras.

29. Mr. H. T. Engstrom: *The determination of the common index divisor of an algebraic field in some general cases.*

It is shown that certain types of prime-ideal decomposition of a rational prime p in an algebraic field of nth degree determine the power of p dividing the common index divisor of the field. An explicit formula for this power of p is found. In the general case, a maximum is established for the common index divisor in terms of the degree of the field.

30. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and unbounded continuous curves.*

The author has previously analysed the relation of bounded continuous curves to the Janiszewski-Mullikin theorem. He now shows that a cyclicly connected unbounded continuous curve satisfying this theorem is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set. He constructs the inverse of the given curve from which it appears that the curve is a generalized cylinder (open at both ends); the curve is called a cylinder-tree, and is shown to be equivalent to a tree of elements which are its simple closed subcurves. The author discusses the case that the unbounded curve is not cyclicly connected.

31. Professor T. R. Hollcroft: *Linear systems of surfaces on a hypersurface in four dimensions.*

A linear system of surfaces on a given hypersurface in four dimensions is defined by the intersections of a linear system of hypersurfaces with the given hypersurface. The investigation includes a study of the properties of the general linear system, complete linear systems, pencils, nets and webs of a linear system, systems of surfaces on a given hypersurface cut out by its polar hypersurfaces. Linear systems of surfaces in ordinary space have been extensively studied. These systems appear as special cases of the above when the order of the given hypersurface is unity. Many of the theorems established for linear systems of three-space surfaces hold also for these more general linear systems.

32. Professor R. G. Putnam: *Note on Riesz's fourth condition for elements of accumulation.*

One of the conditions which elements of accumulation must satisfy in the abstract sets considered by F. Riesz is the following: An accumulation element A of a set E is determined by the subsets of E of which A is an accumulation element. In a Hausdorff space one of the conditions which neighborhoods must satisfy is: To every pair of distinct points x and y there correspond neighborhoods U_x and U_y which are disjoint. This note points out a relation between these two conditions.

33. Mr. Hassler Whitney: *A peculiar function.*

The function considered is continuous at points of an everywhere dense set, and discontinuous in the complementary set. At points of continuity it has no differential coefficient, right- or left-handed, while in the complementary set the differential coefficient is definitely infinite.

34. Dr. T. C. Benton: *A definition of an unknotted simple closed curve.*

A simple closed curve may be defined as a set of points $f(t)$ where t is a parameter ranging over the interval $t_1 \leqq t \leqq t_2$ and $f(t) \neq f(t')$ unless (i) $t=t'$, or (ii) $t=t_1$, $t'=t_2$, or (iii) $t=t_2$, $t'=t_1$. If there is a function $f(t, \lambda)$ (which is uniformly continuous) such that for the range $t_1 \leqq t \leqq t_2$, $0 \leqq \lambda \leqq 1$, we have the following properties: (i) $f(t, 1)=f(t)$, where $f(t)$ is the function defining the curve above; (ii) $f(t, 0)=f_0$, a constant; (iii) $f(t', \lambda')=f(t'', \lambda'')$ if and only if (a) $\lambda'=\lambda''=0$, or (b) $\lambda'=\lambda''\neq 0$ and one of the following holds: (1) $t'=t''$, (2) $t'=t_1$, $t''=t_2$, (3) $t'=t_2$, $t''=t_1$; then the simple closed curve $f(t)$ is an unknotted curve. It is shown that curves which satisfy this definition can be imbedded in a plane, and that those which are imbedded in a plane must be unknotted according to this definition.

35. Dr. W. L. Ayres (National Research Fellow): *Continuous curves homeomorphic with the boundary of a plane domain.*

In this note it is shown that in order that a continuous curve M be homeomorphic with a plane continuous curve which is the boundary of one of its complementary domains it is necessary and sufficient that every maximal cyclic curve of M be a simple closed curve.

36. Professor E. T. Browne: *On the signature of a quadratic form.*

In this paper a study is made of the signature of a (real) quadratic form of rank r in which $A_r \neq 0$ but which is not a regular form since we suppose that two or more consecutive terms vanish in the sequence of A's from which the signature is determined. The method employed is the rearrangement scheme of Frobenius in conjunction with Gundelfinger's rule and furnishes a very simple means of arriving at the results previously obtained in a more complicated manner by Frobenius when two consecutive A's vanish, and giving a result not given by Frobenius when three consecutive A's vanish. It is also shown how the same scheme of rearrangement may be employed in determining the signature of any real recurring form, whether regular or not.

37. Dr. T. H. Gronwall: *The number of arithmetic operations involved in the solution of a system of n linear equations.*

It is shown that, denoting by A_n, M_n, D_n the number of additions (or subtractions), multiplications, and divisions required for solving the system $\sum a_{ik}x_k = b_i$, $(i, k=1, \cdots, n)$, by the method of substitutions, then $A_n = M_n = n(n-1)(2n+5)/6$, $D_n = n(n+1)/2$, for the most advantageous arrangement of the method. The reduction in these numbers when the matrix (a_{ik}) is symmetric is determined, and other methods of solution are discussed.

38. Professor Edward Kasner: *Biharmonic functions and certain generalizations.*

In an earlier paper the author showed that if a function $F(x, y, x_1, y_1)$ is converted by *every* conformal substitution $x_1 = \alpha(x, y)$, $y_1 = \beta(x, y)$ into a harmonic function of x, y, then F must be biharmonic in the sense of Poincaré (real part of an analytic function $f(z, z_1)$ of two complex variables). The same result holds if we use merely the similitude group. If, however, we use smaller groups, for example the translation or rotation groups, larger classes of functions F are obtained, which we may then regard as generalizations of the biharmonic class. The most interesting generalization arises for the group of rigid motions. The substitutions converting an arbitrary function F into a harmonic function do not always form a group, and examples of this sort are also studied. Finally an analogous question for polygenic (non-analytic) functions of two complex variables z, z_1 is discussed: when can such a *polygenic* function, by means of a set of *analytic* substitutions $z_1 = \omega(z)$, be converted into an *analytic* function of z?

39. Professor Norbert Wiener: *Group theory and harmonic analysis.*

The author shows that his methods of generalized harmonic analysis may be extended to analysis in other Eigenfunktionen than the trigonometric functions by substituting for the transformation group $f(x) \rightarrow f(x+t)$ the cyclical transformation group T^λ, where T^λ is asymptotically permutable with the transformation which makes $f(x)$ vanish when $|x| > A$.

40. Professor J. L. Walsh: *Boundary values of analytic functions and the Tchebycheff method of approximation.*

Let $f(z)$ be a function of the complex variable z analytic on the unit circle C: $|z| = 1$, but with singularities interior to this circle. This paper considers approximation to the function $f(z)$ by polynomials $p_n(z)$, when the measure of the closeness of the approximation is $\epsilon_n = \max |f(z) - p_n(z)|$, z on C. It is of course true that ϵ_n cannot approach zero with $1/n$, but ϵ_n nevertheless has (for all polynomials $p_n(z)$ without restriction) a lower limit ϵ. In the simplest cases, for instance if $f(z)$ is a polynomial in $1/z$, there exists a function $p(z)$, analytic for $|z| \leqq 1$, such that $\max |f(z) - p(z)| = \epsilon$, z on C, and it can be shown by the use of certain results due to Carathéodory that $\epsilon_n \rightarrow \epsilon$ implies $p_n(z) \rightarrow p(z)$ uniformly for $|z| \leqq r < 1$. If $p_n(z)$ is the Tchebycheff polynomial of degree n, for approximation to $f(z)$ on C, then we have $p_n(z) \rightarrow p(z)$ uniformly in a region in whose interior C lies. There are extensions to more general curves C and to more general functions $p_n(z)$ and $f(z)$.

41. Dr. Lucien Féraud: *On Birkhoff's Pfaffian systems.*

The Birkhoff Pfaffian systems of differential equations, arising from a variational principle $\delta\int_{t_0}^{t_1}[\sum_{i=1}^{2m}X_i dx_i/dt+Z]dt=0$, include those of Hamiltonian type, and are invariant in form under an arbitrary point transformation. In the present paper most of the classical properties of the Hamiltonian systems are extended to the Pfaffian systems; particular attention is paid to those properties concerning the reduction of the order of the system in case a certain number of integrals are known. A "last multiplier" of simple form is obtained. The transformation of a Lagrangian system into a Pfaffian and also the reduction of the Pfaffian form to the Hamiltonian are studied. The fundamental fact that this last reduction is possible, not only in a *formal* sense but in an *actual* sense, is established for the case of two degrees of freedom.

42. Mr. C. I. Lubin: *A note on singular points of the differential equation* $dx/[X(x,y)]=dy/[Y(x,y)]$.

The real functions $X(x, y)$, $Y(x, y)$ appearing in the differential equation $dx/X=dy/Y$ are supposed to be analytic in the neighborhood of the origin and to vanish there. The determinant made up of the coefficients of the linear terms of the power series expansions of $X(x, y)$ and $Y(x, y)$ is assumed to be different from zero and to have characteristic roots λ_1, λ_2 of opposite signs. This case has not been covered at all in the classical theory, while the recent treatment by Birkhoff by means of divergent series supposes there is no commensurability relation between λ_1 and λ_2. In this note it is proposed to obtain a normal form for the equation when $p\lambda_1+q\lambda_2=0$, where p and q are positive integers without common factor. The author proves that there exists a function $g(x, y)=y+\cdots$ involving one arbitrary constant, analytic for $x\neq 0$ and $y\neq 0$, continuous together with all its partial derivatives for $x=0$ or $y=0$, analytic in x for $y=0$ and analytic in y for $x=0$, which, for $z=x$, $w=g(x, y)$, transforms the equation into the form $dz/(\lambda_1 z)=dw/(\lambda_2 w-Az^{mp}w^{mq+1}-Bz^{2mp}w^{2mq+1})$, where A and B are constants (either or both of which may be zero) depending on the functions $X(x, y)$, $Y(x, y)$.

43. Dr. Caroline E. Seely: *Note on kernels of positive type.*

In an earlier paper (Annals of Mathematics, (2), vol. 20, p. 174) the author showed that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its fundamental functions $\Phi_i(s)$, $\Psi_i(s)$, then $\int_a^b\Phi_i(s)\Phi_j(s)ds=0$, $\int_a^b\Psi_i(s)\Psi_j(s)ds=0$, $\lambda_i\neq\lambda_j$. In the present note it is shown by an example that this condition is not necessary in order that $K(s, t)$ itself be of positive type with respect to every function developable in a Fourier series.

44. Professor H. S. Vandiver: *On the first case of Fermat's last theorem.*

In this paper the author examines the equation $x^l+y^l+z^l=0$, where

x, y, and z are rational integers prime to the odd prime l and none zero. The special case $x \equiv y \pmod{l}$ is discussed, and among other results the following theorem is obtained: If the above equation is satisfied under the conditions mentioned, and $x \equiv y \pmod{l}$, then there exists no prime integer in the set $1+2l$, $1+4l, \cdots, 1+(l-1)l$. This paper will appear in the Annals of Mathematics.

45. Professor H. S. Vandiver: *A theorem of Kummer's concerning the second factor of the class number of a cyclotomic field.*

This paper has appeared in full in the May-June number of this Bulletin.

46. Professor H. S. Vandiver: *An algorithm for transforming the Kummer criteria in connection with Fermat's last theorem.*

In this paper the author applies the theory of power characters in the field $k(\zeta)$, $\zeta = e^{2i\pi/l}$, l an odd prime, to the equation $x^l+y^l+z^l=0$, where x, y, and z are rational integers prime to l and none zero. By means of this method various combinations of the well known Kummer criteria for the solution of this equation are obtained, including new derivations of the congruences $2^{l-1} \equiv 3^{l-1} \equiv 1 \pmod{l^2}$. This paper will appear in the Annals of Mathematics.

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