A THEOREM CONCERNING SIMPLY TRANSITIVE PRIMITIVE GROUPS*

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The theorem here presented has evolved by easy stages from a paragraph in Jordan's *Memoir on primitive groups.*† In the discussion of a particular class of simply transitive primitive groups, he showed that the degree of a doubly transitive constituent of the subgroup leaving one letter fixed cannot be greater than the sum of the degrees of all the other transitive constituents.

THEOREM. *Let H be the subgroup that fixes one letter of a simply transitive primitive group. If one of the constituents of H is a doubly transitive group of degree m, there is in H a transitive constituent whose degree is greater than m and divides $m(m-1)$.*

Let G, of degree n and of order nh, be the given simply transitive primitive group. Let H, the subgroup of G that leaves the letter x fixed, be denoted by $G(x)$.

Let it be assumed: (1) that $G(x)$ has exactly k similar‡ doubly transitive constituent groups: A on the letters $a_1, a_2, \cdots, a_m$; B on $b_1, b_2, \cdots, b_m$; $\cdots$; K on $k_1, k_2, \cdots, k_m$; (2) that $G(a_1)$ has $k-1$ doubly transitive constituents: B_1 on $b_1, a_2, \cdots, a_m$; C_1 on $c_1, b_2, \cdots, b_m$; $\cdots$; K_1 on $k_1, j_2, \cdots, j_m$; (3) that n is greater than $km+1$. These assumptions, when $k=1$, reduce to the hypothesis of our theorem. We wish to prove by induction that there is a transitive

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† C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), p. 198, §64.

Manning, American Journal of Mathematics, vol. 39 (1917), p. 298; Transactions of this Society, vol. 20 (1919), p. 66; *Primitive Groups*, 1921, p. 83; Transactions of this Society, vol. 29 (1927), p. 821, §8.

‡ Manning, Transactions of this Society, vol. 29 (1927), p. 821, §8.

constituent of some degree μ in $G(x)$ such that μ divides $m(m-1)$ and is greater than m.

The letters of G are supposed to be so chosen that each substitution of $G(x)$ permutes the m subscripts of the letters of the similar groups $A, B, \cdots, K$ in exactly the same way.

The subgroup $G(x)(a_1)$ of $G(x)$ (that fixes a_1) is of order h/m. The subgroups $G(x)(a_1)$ and $G(a_1)(x)$ are identical, and therefore in $G(a_1)$ x is a letter of a transitive constituent of degree m. $G(a_1)(x)(=G(b_1)(x)=G(c_1)(x)= \cdots =G(k_1)(x))$, because of assumption (1) has k transitive constituents of degree $m-1$ each on the letters $a_2, a_3, \cdots, a_m; b_2, b_3, \cdots, b_m; \cdots; k_2, k_3, \cdots, k_m$, respectively. Since G is primitive, $\{G(x), G(a_1)\}=G$, and therefore $G(a_1)$ cannot have a transitive constituent of degree $m-1$ on the letters $k_2, k_3, \cdots, k_m$, nor a transitive constituent of degree m on the letters $x, k_2, \cdots, k_m$. In $G(a_1)$ the letters $k_2, k_3, \cdots, k_m$ all belong to the same transitive constituent of unknown degree $\mu(\geqq m)$. Then $G(a_1)(k_2)$ is of order h/μ and if x belongs to a transitive constituent of degree $\delta(\geqq 1)$ in $G(a_1)(k_2)$, $G(a_1)(k_2)(x)$ is of order $h/\mu\delta$. Then since $G(x)(a_1)(k_2)=G(x)(k_1)(k_2)$, $h/\mu\delta = h/[m(m-1)]$, and μ divides $m(m-1)$. Either our theorem is proved or $\mu=m$. If $\mu=m$, $G(a_1)$ has this transitive constituent (L_1) on the letters $l_1, k_2, \cdots, k_m$ (where $l_1 \neq x$); L_1 is doubly transitive because it contains a transitive subgroup of degree $m-1$ on $k_2, k_3, \cdots, k_m$.

The substitution $(a_1 a_i \cdots)\cdots$ of $G(x)$ transforms $G(a_1)$ into $G(a_i)$ and L_1 into L_i, a doubly transitive constituent of $G(a_i)$ on the m letters $l_i, k_1, k_2, \cdots, k_{i-1}, k_{i+1}, \cdots, k_m$ $(l_i \neq x)$. Nor is $l_i = l_1$, for if it were, $\{G(a_1), G(a_i)\}$ would have a transitive constituent of degree $km+1(<n)$ on the letters $a_1, a_2, \cdots, k_m, l_1$. Then $l_1, l_2, \cdots, l_m$ are the letters of a transitive constituent L of $G(x)$. Any substitution of $G(x)$ that replaces a_1 by a_i replaces l_1 by l_i, so that the substitutions of $G(x)$ permute the letters of A and the letters of L in exactly the same way. The groups A and L are similar.

Suppose now that $n=(k+1)m+1$. Then $G(a_1)$ has a transitive constituent X_1 on the m letters $x, l_2, \cdots, l_m$. $G(a_1)(x)$

$(=G(b_1)(x)= \cdots =G(l_1)(x))$ has $k+1$ transitive constituents on $a_2, a_3, \cdots, a_m$; $b_2, b_3, \cdots, b_m$; $\cdots$; and $l_2, l_3, \cdots, l_m$. No two of the groups $G(b_1)$, $G(c_1)$, $\cdots$, $G(l_1)$ can have the transitive constituent $a_2, a_3, \cdots$ in common. Because B_1 (on $b_1, a_2, \cdots, a_m$) is in $G(a_1)$, in one of these k groups x, $a_2, \cdots, a_m$ are the letters of a doubly transitive constituent. Let these subgroups be transformed by $(x)(a_1a_2 \cdots) \cdots$ into $G(b_2)$, $G(c_2)$, $\cdots$, $G(l_2)$. One of the transformed groups has a doubly transitive constituent on the letters $x, a_1, a_3, \cdots, a_m$, and therefore contains a substitution $S=(xa_1) \cdots$ which permutes the letters $a_3, a_4, \cdots, a_m$ among themselves. Then S should transform the constituent A of $G(x)$ into the constituent X_1 of $G(a_1)$; but this, because $m \geqq 3$, S cannot do. Hence $n>(k+1)m+1$. We have shown too that $G(x)$ has $k+1$ doubly transitive constituents $A, B, \cdots, L$ whose letters are permuted by $G(x)$ in exactly the same way, and that $G(a_1)$ has k transitive constituents $B_1, C_1, \cdots, L_1$. Thus the three conditions of our assumption are reproduced with $k+1$ replacing k.

Since n is finite, this process can lead only to the conclusion that the degree of some transitive constituent of $G(x)$ is greater than m and divides $m(m-1)$.

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