THE EQUATION OF THE V_{n-1}^{n-1} IN S_n*

BY C. A. RUPP

Segre, dealing with synthetic and enumerative geometry, has twice mentioned† that the manifold formed by all the lines of S_n which meet n given S_{n-2}'s is a V_{n-1}^{n-1}. In a recent article, Wong‡ mentions that the equation of the general V_{n-1}^{n-1} is unknown. It is the purpose of this paper to exhibit the equation.

The method is the following: through a general point of one of a set of n given S_{n-2}'s is passed a line which is required to meet each of the remaining $(n-1)$ S_{n-2}'s. The eliminant of the system of equations thus set up is the desired equation, as will shortly appear.

Let the equations of the n given S_{n-2}'s be

$$x_i = 0 = \sum_{j=0}^{j=n} a_{ij}x_j, \qquad (i = 1, 2, \cdots, n), \qquad (a_{ii} = 0).$$

Berzolari§ has shown that the above display does not particularize the set of S_{n-2}'s, but is a mere question of a suitable choice of the reference system. We shall need the Grassmann-Plücker coordinates of the S_{n-2}'s, that is, the two-rowed determinants from the matrices of the coefficients in their equations. In each set of coordinates we here find $n(n-1)/2$ elements are zero; the remaining n are some of the numbers a_{ij}, prefaced with a proper sign.

Let y be the coordinates of a point on one of the S_{n-2}'s, say the first one. We then have that

$$\text{(A)} \qquad y_1 = 0, \qquad \sum_{j=0}^{j=n} a_{1j}y_j = 0.$$

* Presented to the Society, December 27, 1928.

† Segre, *Mehrdimensionale Räume*, Encyklopädie, vol. III C7, pp. 815 and 832.

‡ Wong, B. C., this Bulletin, vol. 34 (1928), pp. 553–554.

§ Berzolari, Rendiconti del Circolo Matematico di Palermo, vol. 29 (1905), p. 229.

Let x be the coordinates of a point on a line through y, and distinct from y. The coordinates of this line may be denoted by

$$p_{ij} = x_i y_j - x_j y_i, \qquad (i,j = 0,1,\cdots, n), \ (i \neq j).$$

The condition that this line meet an S_{n-2} is that the summation of the product of corresponding coordinates vanish. When we stipulate that the line shall meet all the S_{n-2}'s of the set, we adjoin to equations (A) the $n-1$ equations (B), where

$$\text{(B)} \qquad \sum_{j=0}^{j=n} a_{ij} p_{ij} = 0, \qquad (i = 2,3, \cdots, n).$$

When we eliminate y from the two sets of equations, the resulting equation in x represents the equation of the V_{n-1}^{n-1}, for the line from x to y has been made to meet each of the n S_{n-2}'s and is hence a generator of the V_{n-1}^{n-1}. One may readily verify that the final form of the eliminant does not depend upon the particular member of the set of S_{n-2}'s whereon one chooses to place the point y.

If the interested reader will consider the following S_3's in S_5,

$$x_0 = x_1 = 0, \quad x_2 = x_3 = 0, \quad x_4 = x_5 = 0, \quad x_0 = x_2 = x_4, \quad x_1 = x_3 = x_5,$$

locate the point y on one of the S_3's, compute the coordinates of the line from x to y as well as the coordinates of the remaining four S_3's, write down the four incidence relations that cause the line to meet the four S_3's, eliminate y from the set, and reduce, he will have the pleasure of finding the equation of a V_4^4 in S_5; the result is

$$0 = x_0x_1x_2x_5 + x_0x_3x_4x_5 + x_1x_2x_3x_4 - x_0x_1x_3x_4 - x_0x_2x_3x_5 - x_1x_2x_4x_5.$$

The Pennsylvania State College