THE HEROIC AGE OF GEOMETRY*

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The remarks which I shall have the honor to make today are addressed primarily, if not exclusively, to the older generation. The prophet Joel describes the golden age to come by saying "Your old men shall dream dreams, and your young men shall see visions." That distinction is eternally valid. Youth has visions of the future. Age dreams of the past. Our younger members are facing forward, looking towards the splendid mathematical discoveries that will be made during the coming decades, among which will be included those which they themselves will have made. It stirs the blood and fortifies the courage to feel that one is called upon to contribute in this way to the advancement of science, and that a grateful posterity will recognize what one has done. If the present passion for materialism continues, the number of contributions to mathematical science in the next hundred years may not greatly exceed half a million. Go on, oh younger generation, happy in the thought that the mathematicians of a century hence will, by an unfailing instinct, pick out your own particular contributions from all the rest!

For you who, like myself, belong to the generation that is passing, who have reached the age to dream dreams, I propose a humbler task. I propose that we look backward. The most notable epoch in all the long history of geometry, the heroic age, was almost exactly a hundred years ago. It was the moment when geometric supremacy was passing from France to Germany, a fact which the French geometers were slow to recognize, for Chasles, in his *Aperçu historique des méthodes en géométrie*, regretted that he could not

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speak of the work of the German geometers because he could not read their language. I shall try to show you some of the problems which interested the geometers of that time, and the methods chosen for their solution. I shall try to point out what was most significant in their work, and what was their influence on the progress of science.

Karl Wilhelm Feuerbach was born in 1800, contemporary with Macaulay and von Moltke, and died in 1834, professor at the gymnasium in Erlangen. In 1822, he published in Nürnberg a pamphlet entitled *Eigenschaften einiger merkwurdigen Punkte des geradlinigen Dreiecks*. The original of this, which may have been his doctor's dissertation, is hard to find, but a reprint appeared in 1908. The writer is concerned with the geometry of the triangle. When he finds two notable points he determines their distance, when there are three he determines the center and radius of the circumscribed circle. The methods are entirely elementary. The inscribed and escribed circles seem to attract him particularly, and finally he proves, quite casually, on page 48, that the circle through the feet of the altitudes passes through the middle points of the sides, and is tangent to the inscribed and escribed circles. This is the first known proof of the nine-point or Feuerbach circle theorem, though the author himself missed three of the nine points. Previous writers had found the center and radius of the circle, Bevan giving their determination as a problem in 1804.* The existence of nine points thereon was discovered in 1821 by Brianchon and Poncelet.† It is no disparagement to Feuerbach that he did not see Bevan's article which appeared in an obscure English journal, while the work of the French writers may well have appeared after his own had gone to press.

The fate of his theorem was that of many another;‡

* Leybourn's Mathematical Repository, vol. 1, p. 18.

† Gergonne's Annales de Mathématiques, vol. 11, p. 215.

‡ See Macaulay, *History of the nine-point circle*, Proceedings of the Edinburgh Mathematical Society, vol. 11 (1892).

first it was overlooked, then rediscovered. Steiner announced it without proof in 1828* and found the remaining three points in 1833.† Next Terquem proved it completely in 1842.‡ Salmon proved the tangency part in 1860 as did Casey,§ although they called the circle merely a six-point circle. Since that time there has been a perfect cloudburst of demonstrations. It has been said that the study of the philosophy of Kant has risen almost to the rank of one of the liberal professions; the same is measurably true of proving Feuerbach's theorem. The last adept would seem to be Sawayama, who gave nine demonstrations, all presumably new, in 1911.‖ Perhaps there is an element of absurdity in all this, but can we deny that we owe to Feuerbach the most beautiful theorem in elementary geometry that has been discovered since the time of Euclid?

Steiner's important contributions to elementary geometry during the period that we are considering were by no means limited to proving the Feuerbach theorem. He brought out and developed several of the most important principles connected with the modern geometry of the circle, notably¶ the idea of the power of a point with regard to a circle, the centers of similitude and radical axes. These ideas, though perhaps new to Steiner were not strictly speaking new to geometry, as they had already been presented by Gaultier,** Durande†† and Poncelet.‡‡ But Steiner handled his methods with consummate skill; witness his beautiful solution of Malfatti's problem to construct three circles, each tangent to two sides of a triangle, and to the other two circles, which he gave

* See his Collected Works, Berlin, 1881, pp. 195, 196.

† Ibid., p. 491, Note.

‡ Nouvelles Annales de Mathématiques, vol. 1, p. 197.

§ Quarterly Journal of Mathematics, vol. 4, p. 153.

‖ L'Enseignement Mathématique, vol. 13 (1911).

¶ *Einige geometrische Betrachtungen*, Journal für Mathematik, vol. 1 (1826).

** Journal de l'École Polytecnique, vol. 9 (1813).

†† Annales de Mathématiques, vol. 11 (1820).

‡‡ *Propriétés Projectives des Figures*, Paris, 1822.

without proof, and for which no simple proof was published till that of Hart thirty years later.* Moreover, his proof of the Feuerbach theorem was contained in a paper which was important in another way, for it deals with constructions possible with the aid of a ruler and one circle fully drawn and with given center. This is not, of course, the first attempt to replace the Platonic instruments, ruler and compass. There were, for a long time, two opposing tendencies, one to extend the list of permissible instruments, the other to restrict it. In the latter category we might mention the idea of using a compass with fixed opening, which goes back to Abul Wafa in the 10th century† or the really astonishing work of Mascheroni in proving that every determination of a point that is possible with ruler and compass, is also possible with the compass alone.‡ The idea of using a ruler and given circle was first broached by Poncelet,§ but was carried through independently and in far greater detail by Steiner.

While in France and Germany distinguished mathematicians were doing their best to build up the ancient science of geometry, further East others were equally zealous in what must have seemed uncommonly like an effort to tear it down. The period we are considering included the birth of the non-euclidean geometry, at present a perfectly respectable branch of mathematics, but a heretical doctrine one hundred years ago. The "fons et origo" was skepticism about Euclid's twelfth postulate, that dealing with parallels. Candid persons had felt for a long time that this was not really as self evident as a reputable axiom ought to be, and countless attempts had been made to prove it. Now one of the classical methods of procedure in mathematics is the reductio ad absurdum, and quite naturally it occurred to those who were trying to prove this axiom, to see what would

* Quarterly Journal of Mathematics, vol. 1 (1855).

† See Tropfke, Journal Asiatique, (5), vol. 5 (1855), p. 226.

‡ *Geometria del Compaso*, 1797.

§ *Propriétés Projectives des Figures*, 1822, pp. 187–189.

happen if they replaced it by some other assumption about parallels. This was first tried by Saccheri at the middle of the 18th century, and many others after him. In each case things went well until the writer slid into some unwarrantable assumption in order to show that he was in difficulties when he wasn't. It took six decades from the time of Saccheri, and the genius of Gauss, to see that the parallel axiom was really independent of the others. But Gauss was busy with other matters and did not take the trouble to publish his results, so that the credit for first publicly announcing the non-euclidean geometry goes to a Russian, Lobachevski, who, on February 12, 1826, read before the Physico-Mathematical Society of Kasan a paper entitled *Exposition succincte des principes de la géométrie.** No trace of the manuscript of this address has ever been found, but we know from Lobachevski's later writings what must have been included therein. It is also permissible to doubt that it deserved the adjective succinct. His Russian article on the *Principles of geometry*, which appeared in the Kasanskij Wjestnik, the Kasan News in 1829–30, purports to be an extract from the previous paper, and covers 66 pages. It is very incomplete, and we only get to the root of Lobachevski's ideas by studying his subsequent publications.

Let us hasten to bracket with Lobachevski, the independent discoverer Johann Bolyai, whose *Appendix, Scientiam Absolute Veram Spatii Exhibens* was attached to his father Wolfgang's *Tentamen Juventutem Studiosam in Elementa Matheseos . . . . Introducendi*, which was published at Maros Vasarhely in 1832.

The truly startling thing about these two works is their similarity. It is not to be wondered at that, when Bolyai first saw Lobachevski's work in 1835, he made the natural mistake of believing that it was copied from his own.† We have learnt in a hundred years that there are a good many

* For an account of Lobachevski, see Engel, *Lobatschefskij*, Leipzig, 1898.

† See Stäckel, *Wolfgang and Johann Bolyai*, Leipzig, 1913, pp. 140 ff.

different ways of setting to work to establish the non-euclidean geometries. Kipling has written somewhere:

"There are nine and sixty ways, of constructing tribal lays,
And every single one of them is right."

It is much the same in constructing non-euclidean geometries. It is therefore really astonishing, that Lobachevski and Bolyai showed the following similarities:

1. Both defined parallel lines as the limiting positions of intersecting ones.

2. Both reached at a bound the hyperbolic geometry, overlooking completely the possibility of an elliptic geometry, where there were no parallels.

3. Both introduced at an early stage the horosphere, which is the surface orthogonal to a bundle of parallel lines, bringing out the remarkable fact that on this surface we have the euclidean geometry.

4. Both pointed out that the formulas of spherical trigonometry could be established without the parallel axiom.

5. Both studied the ratio of the non-parallel sides of a quadrilateral, two of whose sides are equal and parallel.

Of course there are divergences as well as similarities in the writings of the two. Lobachevski made continual use of the parallel angle associated with a given distance, that is, the angle which the parallel to a given line, through a point at a given distance from that line, makes with the normal. Bolyai manipulated skillfully his theorem that the ratio of the sines of two angles of a triangle is that of the circumferences on the opposite sides as diameters. Lobachevski is more prolix, and especially interested in trigonometric formulas, Bolyai is brief, but he brings out the function of the space constant in admirable fashion. My own final impression is that the points of similarity are far more remarkable than those of difference.

It was many years before the significance of non-euclidean geometries was fully understood, and developed beyond the point reached by the discoverers. It was only in 1854 that the third classical geometry, the elliptic geometry of Riemann

was first exhibited.* Skepticism lingered; in order to convince the average man, or the average mathematician, it was necessary to see the non-euclidean geometry at work. A sphere, or rather a hemisphere where opposite points of the equator count as identical, gives a very good elliptic plane; a beautiful specimen of a Lobachevskian plane, or rather of a part of one, was found by Beltrami in a surface of constant negative curvature.† These examples did not go beyond two dimensions. Klein seized on the theory of projective measurement, casually thrown out by Cayley‡ to give a perfect example of Lobachevskian or Riemannian geometry in as many dimensions as may be desired. Lie's theory of continuous groups threw a new and valuable light on the questions involved; contributions came in from various quarters. Since the beginning of the present century the subject has been somewhat transcended, owing to the newer methods of differential geometry, and the theory of relativity. Of the various methods of attack, that of Riemann based on the study of quadratic differential forms has shown itself best able to meet the new demands. But we owe an enormous debt to Lobachevski and Bolyai, not only for enriching geometry, but for initiating a movement which has been of incalculable importance to philosophy. Our whole modern conception of mathematics as a logical system based on arbitrary axioms may be traced back to their pioneer work.

We have had occasion already to speak of Steiner once or twice; we have not yet said a word about his most important work, *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*,§ one of the corner stones on

* *Ueber die Hypothesen welche der Geometrie zur Grunde liegen.* See his Collected Works. 2d ed., Leipzig, 1892, pp. 272 ff.

† *Saggio di interpretazione della geometria non-euclidea*, Giornale di Matematiche, vol. 6 (1868).

‡ Klein, *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 4 (1871); Cayley, *Sixth Memoir on Quantics*, Philosophical Transactions, vol. 149 (1859).

§ Berlin, 1832. See his Collected Works, vol. 1.

which the whole subject of synthetic projective geometry rests. Who were the predecessors, from whom, presumably, Steiner drew inspiration? They were certainly French. Among the earlier writers there were, of course, Pascal, Desargues and Brianchon. Then Carnot, although preceding the modern movement, exercised considerable influence through his *Géométrie de Position*,* especially through his theory of transversals. More important by far was Poncelet, whose *Propriétés Projectives des Figures*, composed in a Russian prison in 1813, first saw the light in 1822. Poncelet's root idea may be stated about as follows. A geometric figure may be wonderfully simplified by a central projection from plane to plane, or by a homology in plane or in space. What are the properties of figures which are invariant under such a transformation, how can theorems about the simplified figures be stated in invariant terms so as to give properties of the original figures? We see at once that the essentials consist in concurrence, collinearity, coplanarity and cross ratios. Poncelet also introduced a rather cumbrous system of what he called "ideal chords" determined not by the given curve, but by another, to get rid of imaginary intersections. Finally he laid stress on the method of polar reciprocation with regard to a conic or quadric to establish the principle of duality. This last principle was a good deal talked about at the time; a long and not very edifying dispute between Gergonne and Poncelet as to priority of discovery is spread over the pages of the Annales de Mathématiques.† The former conceived duality in a broader spirit as an inherent characteristic of geometric figures, but did not follow this important idea very far; the latter did not much believe in any duality he did not see, so he created it by polarization.

All of these threads were drawn together by Steiner in his development of projective geometry, which is nothing if not "systematisch." He starts with a point, the points

* Paris, 1803.

† Volume 18 (1827–28).

of a line, and those of a plane. The plane is the simplest two-dimensional locus of points, the point the simplest two-dimensional envelop of planes, the line is the simplest one-dimensional figure of points or planes, the pencil of lines is a self-dual figure. The principle of duality appears at the very start, and a good part of the work is printed in double columns. Cross ratios of points, lines and planes are then defined, and it is shown how they are unaltered by projections and intersections. Steiner talks a good deal about "projective forms" but the definition is rather mixed with theorems about them. The underlying thought is that they are one-dimensional forms whose elements are in one-to-one correspondence, with equality of corresponding cross ratios. Steiner shows easily enough that two forms which are connected by a finite number of projections and intersections are projective in this sense; his proof of the converse is faulty owing to his inability to handle the question of the continuity of the projective relation. Starting with these data, Steiner obtains all the other figures he wants to study by construction; a conic is given by the intersections of corresponding lines in two projective pencils, and enveloped by lines connecting corresponding points in two projective ranges.

Every modern student of projective geometry will see that the fundamental ideas of his science are here set out in order. Steiner laid a very solid foundation whereon his successors might build. It is true that the Latin school of geometers, including such mathematicians as Chasles and Cremona, drew their inspiration more from Poncelet than from Steiner, but the latter had a notable lineal descendant in von Staudt. This profound thinker perceived two flaws in the perfection of the Steinerian structure. In the first place, although cross ratios are projectively invariant, their original definition is metrical; secondly, there is no satisfactory treatment of imaginary elements. These defects he set out to remedy in heroic fashion.* Each set of four ele-

* See his *Geometrie der Lage*, 1847, and *Beiträge zur Geometrie der Lage*, 1856–1860.

ments of a fundamental one-dimensional form he calls a "throw" and associates with a number; four harmonic elements, in right order, are associated with the number -1. He then defines the sum and the difference of two throws, and shows how a throw can be found corresponding to every rational number. At that point he was blocked as he had no means of handling irrational throws, no Dedekind cut. The lack was later filled by Klein.* Von Staudt's other highly original idea was to define an imaginary point as an elliptic involution of points, to which a sense of description has been attached. He showed how projective geometry, enlarged by these new elements, followed the old laws and permitted the old constructions. His culmination is the treatment of a complex throw.

The synthetic geometry of Poncelet and Steiner was vigorously pursued for a century, the last heroic figure being that of Reye. In recent years we have tended to question the wisdom of too sharp a separation between synthetic and analytic methods, and to confine our researches either to tinkering with the fundamental assumptions, or to the field of projective differential geometry. But the synthetic methods have a compelling charm when rightly presented, and afford a most admirable training for every geometer, wherever his specialty may lie.

The great progress of synthetic geometry in the years we have been considering was, fortunately, not at the expense of analytic geometry; on the contrary, the rivalry between the two was of great value to mathematical science. The progress of algebraic methods kept pace with that of the synthetic ones; strangely enough the first writer we must mention in this new connection is our previous acquaintance, Karl Wilhelm Feuerbach. In 1827 we find him publishing his *Grundriss zur analytischen Untersuchung der dreieckigen Pyramide.* His object here is analogous to that previously pursued in the study of the triangle, to establish relations

* Mathematische Annalen, vol. 7.

between the distances between the notable points of the pyramid. But he now passes from synthetic to analytic methods, and consequently arrives at very different theorems from those reached before. To begin with, and most important, he sets up a novel set of coordinates. He has the idea of linear dependence firmly fixed in his mind. Suppose that the coordinates of a variable point are expressed as linear combinations of those of four fixed non-coplanar points

$$X = ax + bx' + cx'' + dx''',$$
$$Y = ay + by' + cy'' + dy''',$$
$$Z = az + bz' + cz'' + dz''',$$
$$1 = a + b + c + d.$$

The quantities a, b, c, d connected by the last identity may be taken as a new system of coordinates. They may be interpreted as the distances to the four faces of a tetraedron, with right algebraic signs, divided by the corresponding altitudes. A linear equation among them will give a plane. Feuerbach has an additional tool in an identity which he attributes to Lagrange, but which I am ashamed to say I have not been able to find in his writings. If A, B, C, D, and E be five points in space, their distances are connected by the relation

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 & 1 \\ BA^2 & 0 & BC^2 & BD^2 & BE^2 & 1 \\ CA^2 & CB^2 & 0 & CD^2 & CE^2 & 1 \\ DA^2 & DB^2 & DC^2 & 0 & DE^2 & 1 \\ EA^2 & EB^2 & EC^2 & ED^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

He proves a good many theorems with the aid of his new tools; some are interesting, others unimportant. Perhaps his most notable result is to give in very simple form the equation of a sphere tangent to four given spheres. It must not be forgotten that Feuerbach never heard of a determinant.

If we must give Feuerbach full credit for the independent discovery of one form of tetrahedral coordinates, we must hasten to point out that he was not the only discoverer. A more important mathematician than he made the same discovery in the same year, August Ferdinand Möbius.* The idea of this able geometer was to base geometric theorems on the properties of the center of gravity of a system of points. If two points be given, any point of their line will be the center of gravity of the two, provided they be endowed with proper masses, positive or negative. These masses are taken as homogeneous coordinates for the point. Two masses whose sum is zero are made to correspond to the infinite point of the line. A similar definition, when three non-collinear points are given, will give the homogeneous coordinates of a point in their place while four non-coplanar points will yield the homogeneous barycentric coordinates of a point in space, exactly proportional to the coordinates of Feuerbach. In representing a straight line, Möbius uses linear dependence on two points thereof, while a plane is given by linear dependence on three. This parametric method yields, of course, the simplest means of handling linear problems. When it comes to studying a curve, in plano or in spatio, Möbius represents the coordinates of a point thereof as rational functions of a given parameter, while two parameters are used for a surface. As the coordinates used are homogeneous, he replaces rational functions by polynomials, whose degree gives the degree of the curve or surface. He knew too much to imagine that all algebraic curves and surfaces could be expressed in this way, but he confined himself to rational varieties, and in particular to quadratic ones. A notable feature is that he has a clear grasp on the fundamental idea of an invariant, that is to say, a property of a geometric figure which is unaltered by the transformations of a certain group. He studies successively motions which keep distances unaltered, affine transformations where

* *Der barycentrische Calcul*, Leipzig, 1827.

areas or volumes are altered in a fixed ratio, and collineations which he defines as transformations which carry collinear or coplanar points into collinear or coplanar ones. His fundamental relations here are the cross ratios of four collinear points, triangle ratios involving six coplanar ones, and tetrahedral ratios involving eight non-coplanar ones. If four points of a plane, no three collinear, be invariant for a collineation, all points of their net of rationality are invariant. Möbius assumes that the transformation must therefore be the identity. He expresses a collineation by a linear transformation of his homogeneous barycentric coordinates; he grasps the principle of duality both in the form of Gergonne, and that of Poncelet.

It would be a mistake to imagine that Möbius' contributions to geometry were limited to the publications of the *Barycentrischer Calcul*, and contemporary articles dealing with the same subject. We are indebted to him for the discovery of the null system in space, and a geometric theory of circle transformations in the plane which is useful in the study of the simplest functions of a complex variable. The form of the *Barycentrischer Calcul* is not exactly that which we should choose today, but the wealth of fruitful ideas is remarkable.

It would certainly seem that Feuerbach and Möbius were enough to share the credit for discovering tetrahedral or trilinear coordinates; but such is not the case. Omitting independent discoveries of a somewhat later date, we must now render full credit to Julius Plücker for his work of 1828.* He started out with the deliberate intention of showing that all of the beautiful results which Poncelet and Steiner had reached by synthetic methods were easily obtained by algebraic analysis. His interest in trilinear coordinates was much less than that of the previous writers mentioned; they appeared to him merely as a sort of abridged notation. Thus if A and A' be the expressions for the distances of a point

* *Analytisch-geometrische Entwickelungen*, vol. 1, 1828; vol. 2, 1832.

from two given lines, the equations $A=0$, $A'=0$ and $A+\nu A'=0$ will represent respectively the first line, the second line or an arbitrary line through their intersection. An almost identical point of view was contemporarily, though doubtless independently, expressed by Bobilier,* to whom we owe also the equation of the first polar of a point with regard to a given curve.† This idea of using a single letter to replace a whole polynomial gave Plücker an easy method of proving not only straight line theorems, but a number of beautiful properties of circles that had been found synthetically by the French geometers. When it comes to conic sections, he uses Cartesian coordinates, in his first volume, giving a good deal of attention to change of axes, and polar reciprocation.

Between writing the first and the second volume, Plücker became very much impressed with the idea of duality, so that Volume 2 is entirely given to line geometry in a plane. The coefficients of a line are the ratios of the coefficients in its equation when expressed in terms of homogeneous Cartesian point coordinates. A large part of the book is given to curves of the second class. The domain is real, so that an ellipse can be defined as a curve of the second class which has a tangent pointing in every direction. The volume, like its pre ecessor, ends with polar reciprocation.

One has the impression, on the whole, that Plücker's work at this moment was less original than that of Möbius, though easier to read, and yielding more results. It is also to be remembered that his greatest contributions to algebraic geometry came later. We owe to him the coordinates of a line in space, which, in the perfected form devised by Klein, are highly important in that branch of science vaguely called "Higher Geometry." We are also indebted to Plücker for the beautiful identities connecting the numbers of point and line singularities of an algebraic plane curve. Let us mention in passing, that we do not surely know to this day

* Annales de Mathématiques, vol. 18 (1827–28).

† Ibid., vol. 19 (1828–29), p. 108.

when a set of solutions of these equations in positive integers necessarily corresponds to an existing curve.

It would certainly seem that with the names of such geometers as Poncelet, Steiner, Möbius, and Plücker, we might close the list of those who contributed to geometric science in the second period just one hundred years ago; it would seem so, and we should be justified, were it not for the the fact that there lived in the small but important town of Göttingen, a man by the name of Johann Carl Friedrich Gauss who published his *Disquisitiones Generales circa Superficies Curvas* in 1827.* This work is not only the firm basis on which all the mass of subsequently discovered differential geometry rests, but is also the most important individual contribution ever made to that branch of mathematical science.

What were the most novel and fruitful ideas contained in Gauss' geometric work? To begin with, and most important, Gauss was the first writer to make consistent use of the method of parametric representation of a surface, that is to say, the method of expressing the Cartesian coordinates of the points of a surface as functions of two independent variables. The idea was not absolutely new. Euler used it in approaching the general problem of the applicability of one surface to another, his exact words being:†

"Et quia per naturam superficiarum quaelibet coordinata debet esse functio binarum variabilium a se invicem non pendentes." Unfortunately this important idea did not bear immediate fruit. There was much interest in differential geometry at the beginning of the nineteenth century, especially on the part of the French mathematicians. Writers like Monge and Dupin who made really important contributions, regularly expressed z as a function of x and y, Cauchy equated a function of the three variables to zero.

* Commentationes Societatis regiae Gottingensis, vol. 6 (1828); vol· his Collected Works, vol. 4, Göttingen, 1873. An English translation with notes by Thompson was published in Princeton, N. J., 1902.

† Leonardi Euleri Opera Posthuma, vol. 1, St. Petersburg, 1862, p. 494.

The superiority of the Euler-Gauss method is so evident that it "jumps to the eyes." The modern developments of differential geometry would never have been possible without it. Gauss first states it in his *Theoria Attractionis* of 1813.* He uses it also in his prize memoir presented to the Scientific Society of Copenhagen in 1822,† but it is in the *Disquisitiones* that it is first convincingly set forth.

A second important feature of Gauss' treatment is the use made of spherical representation. Gauss considers not one surface but two. Previous writers paid attention to the direction cosines of the normals; Gauss treats these as the Cartesian coordinates of a point on an auxiliary sphere. The beauty of the conception can be seen in the following way. In the plane we may associate with each curve a circle of unit radius whose points correspond to the oriented normals of unit length of the given curve. The curvature is the ratio of the lengths of corresponding infinitesimal arcs of circle and curve. In the same way the total curvature of a surface is the ratio of corresponding infinitesimal surface elements on sphere and surface. It was in this fashion that Gauss presented his idea; it had been given in substance, if not in so many words, by Olinde Rodrigues,‡ who wrote the equation

$$\frac{1}{RR'} = \left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial Y}{\partial y}\right) - \left(\frac{\partial X}{\partial y}\right)\left(\frac{\partial Y}{\partial x}\right).$$

Still the credit for actual statement is Gauss', the year 1816.§

The important part of Gauss' work on the measure of curvature is not in the statement given, but in the demonstration that this expression is invariant under every transformation of the surface which leaves distances invariant, so that mutually applicable surfaces have, at corresponding points, the same measure of curvature. Moreover, this

* Collected Works, vol. 5, 1867, p. 14.

† Ibid., vol. 4, pp. 189 ff.

‡ Correspondence de l'École Royale Polytechnique, vol. 3, 1816, p. 168.

§ Works, vol. 8 (1900), p. 367.

particular invariant, and its extension to higher spaces, has proved to be fundamental in non-euclidean geometry, as pointed out by Riemann and Beltrami and many others.

A good proportion of the *Disquisitiones* is devoted to the study of geodesic or shortest lines. Gauss reaches the differential equations for such lines by straightforward methods of the calculus of variations. His equations show at once the characteristic property that the principal normal to the curve is normal to the surface, a fact first discovered by Johann Bernoulli.* What is rather curious is that Gauss does not point out that geodesic lines may be characterized by the fact that geodesic curvature is everywhere zero, even though Gauss himself appears to have been the first to study this sort of curvature.† He presently proves the existence of geodesically parallel curves. The latter part of the essay is largely devoted to the study of geodesic triangles.

If the ultimate influence of the *Disquisitiones* was incalculable, we can not affirm that its importance was immediately felt.‡ He had an immediate and devoted follower in Minding, who published an entirely Gaussian article on the development of curves on surfaces in 1830.§ But in 1831, we find, Mlle. de St. Germain writing¶

"Si par rapport aux surfaces on avait besoin de connaître la mesure de courbure . . . le mémoire d'Euler contient tout ce que l'on sait d'important à cet égard." The surprising thing is not that a French woman should write this, but that Crelle's Journal should publish it.

It is fair to say that Mlle. de St. Germain will have nothing to do with any measure of curvature but mean curvature. It seems unlikely that the ideas of Gauss were understood or appreciated at all in France till the indefatigable

* See his letter to Leibnitz in Gott. Gul. Leibnitii et Johann Bernoullii Commercium, vol. 1, Lausanne, 1745, p. 393 near bottom.

† See his fragment on *Seitenkrummung*, Works, vol. 8, pp. 386 ff.

‡ In this matter I cannot agree with Stäckel, *Gauss als Geometer*, Gauss' Collected Works, vol. 10, Part II, Section IV.

§ Journal für Mathematik, vol. 6.

¶ Ibid., vol. 7.

Liouville lectured on them in the autumn of 1850.* Perhaps we may say that the final consecration only came in 1867 when Bonnet† proved that every set of solutions of the Codazzi equations determines a single surface completely, except for motions of space.

Such was the work of the leading geometers of one hundred years ago. What shall we say of them today? Well, for my part, I am willing to quote Scripture and say "There were giants in the land in those days." Think of Poncelet in a Russian military prison at Saratoff laying down the fundamental principles of projective geometry, Lobachevski and Bolyai breaking the chains of servitude to Euclid that had lasted for twenty-one centuries, Gauss laying an absolutely secure foundation for the geometry of any manifold in any space. Such work calls for ability of a high order. Moreover, there is one common element in all this work to which I should like to call your attention, the very nice balance between specific results and general theory. A distinguished Scots philosopher, whose lectures I once had the privilege of attending, was forever insisting that it was the great task of philosophy to "Combine the univairsal and the parteecular in a higherrr uneetie." It seems to me that geometry is called upon to do much the same thing. No individual theorem, be it that of Feuerbach, or even that of Pythagoras, can rise to the dignity of a general mathematical principle. Every mathematical proposition or system must be given the greatest possible breadth, in reason. But I wonder whether we are not today in some danger of extinguishing the vital spark in geometry by the excessive abstraction and generality of the results which we seek and publish. It is my personal credo that geometry is a branch of art. I find more emotional appeal in a Raphael Madonna or a Gothic cathedral than I could ever get from a picture of "Things in general" or a building intended for all possible purposes.

Perhaps the difficulty is inevitable, and there remain no

* See Liouville, Comptes Rendus, vol. 32 (1851), p. 533.

† Journal de l'École Polytechnique, vol. 25, Cahier 42, pp. 31 ff.

geometric results to be found which are at once so definite and yet so general as what has been found in the past. This may be so, but I like to think otherwise. I do not so far despair of the Republic. If anyone had asked Lagrange, who died but fifteen years before the beginning of the epoch we have considered, whether he believed there was much yet to do in geometry, he might well have given a hesitant answer. Let us have confidence that our five-thousand-year old science of Earth-measurement has still some beautiful secrets which she will yield when challenged by anyone who is sufficiently able and sufficiently fearless. Let us believe that

> "Some noble deed of note may yet be done
> Not ill-befitting men that strove with gods."

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