CANONICAL CONFIGURATIONS ASSOCIATED WITH A SURFACE*

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1. *Introduction.* After summarizing briefly for subsequent use some of the theory of the projective differential geometry of a surface referred to its asymptotic net, we next present some very convenient formulas for differentiating the coordinates of a point referred to the local tetrahedron customarily employed when the differential equations of the surface are in Fubini's canonical form. Their use is first illustrated in connection with certain configurations generated by different parts of the local tetrahedron at a variable point on the surface, and then they are applied in a brief study of the first and second canonical surfaces, the former being enveloped by the canonical plane and the latter being generated by the canonical point. Finally, the general theory of a pencil of conjugate nets on a surface is applied to a new covariant pencil of conjugate nets, and a sequence of such pencils is constructed.

2. *Analytic Foundations.* In ordinary projective space the four homogeneous coordinates $x^{(1)}, \cdots, x^{(4)}$ of a point P on an analytic non-ruled surface S are functions of two independent variables u, v. If the asymptotic net is parametric and if the proportionality factor of the coordinates is suitably chosen, the functions x satisfy two partial differential equations in Fubini's canonical form

$$(1) \qquad \begin{cases} x_{uu} = px + \theta_u x_u + \beta x_v, \\ x_{vv} = qx + \gamma x_u + \theta_v x_v, \end{cases} \qquad [\theta = \log(\beta\gamma)].$$

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The most general transformation of proportionality factor and parameters preserving this canonical form is

$$(2) \qquad x = \text{const.}\ \bar{x}, \qquad \bar{u} = U(u), \qquad \bar{v} = V(v).$$

A point y given by an expression of the form

$$(3) \qquad y = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$$

is said to have local coordinates $x_1, \cdots, x_4$ referred to the local tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point. The vertices of this tetrahedron are covariant points; in fact the line $x x_{uv}$ is the projective normal and the line $x_u x_v$ is the reciprocal polar of this line with respect to the quadric of Lie; the vertex x_{uv} is the point where the tangent of the curve $u = \text{const.}$ on the surface generated by the point x_u meets the projective normal. The effect of (2) on the local coordinates is the change of unit point given by

$$(4) \quad \bar{x}_1 = x_1, \quad \bar{x}_2 = U' x_2, \quad \bar{x}_3 = V' x_3, \quad \bar{x}_4 = U' V' x_4,$$

accents indicating differentiation with respect to the variables of the functions concerned.

The equation of the canonical plane of S at P is

$$(5) \qquad\qquad \phi x_2 - \psi x_3 = 0,$$

where

$$(6) \qquad \phi = \frac{\partial}{\partial u} \log \beta \gamma^2, \qquad \psi = \frac{\partial}{\partial v} \log \beta^2 \gamma.$$

And the coordinates of the canonical point are $(0, \psi, -\phi, 0)$.

3. *The Derivatives of the Local Coordinates.* We shall first give some *formulas for the local coordinates of the two first-order derivative points of the point y given by* (3). By easy calculation we find

$$(7) \qquad \begin{cases} y_u = y_1 x + y_2 x_u + y_3 x_v + y_4 x_{uv}, \\ y_v = z_1 x + z_2 x_u + z_3 x_v + z_4 x_{uv}, \end{cases}$$

where

$$
(8)
\begin{cases}
y_1 = x_{1u} + p x_2 + (p_v + \beta q) x_4, \\
y_2 = x_{2u} + x_1 + \theta_u x_2 + (\beta\gamma + \theta_{uv}) x_4, \\
y_3 = x_{3u} + \beta x_2 + \pi x_4, \\
y_4 = x_{4u} + x_3 + \theta_u x_4, \\
z_1 = x_{1v} + q x_3 + (q_u + \gamma p) x_4, \\
z_2 = x_{2v} + \gamma x_3 + \chi x_4, \\
z_3 = x_{3v} + x_1 + \theta_3 x_v + (\beta\gamma + \theta_{uv}) x_4, \\
z_4 = x_{4v} + x_2 + \theta_v x_4,
\end{cases}
$$

the functions π, χ being defined by

$$
(9) \qquad \pi = p + \beta\psi, \quad \chi = q + \gamma\phi.
$$

It occasionally happens that we have the equation in local coordinates of a surface covariantly connected with a point P of S and wish to find the envelope of this surface as P varies on S. The usual method of procedure is to differentiate partially with respect to u and with respect to v in the equation and then to solve the two resulting equations with the original to obtain the points of contact of the surface with its envelope. We shall now obtain the differentiation formulas to be used in such a situation.

The coordinates of a point $\bar{x}$ on S near P_x are given* by the expansions

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(10) \quad
\begin{cases}
x_1 = 1 + \cdots, & x_2 = \Delta u + \cdots, \\
x_3 = \Delta v + \cdots, & x_4 = 0 + \cdots.
\end{cases}
$$

And the coordinates y, z, w of the corresponding points $\bar{x}_u$, $\bar{x}_v$, $\bar{x}_{uv}$ near x_u, x_v, x_{uv} respectively are given in order by the power series

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* Stouffer and Lane, *Recent developments in projective differential geometry*, this Bulletin, vol. 34 (1928), pp. 453–472. See part B, equation 2.

$$(11) \quad \begin{cases} y_1 = p\Delta u + \cdots, & y_2 = 1 + \theta_u \Delta u + \cdots, \\ y_3 = \beta\Delta u + \cdots, & y_4 = \Delta v + \cdots; \end{cases}$$

$$(12) \quad \begin{cases} z_1 = q\Delta v + \cdots, & z_2 = \gamma\Delta v + \cdots, \\ z_3 = 1 + \theta_v\Delta v + \cdots, z_4 = \Delta u + \cdots; \end{cases}$$

$$(13) \quad \begin{cases} w_1 = (p_v + \beta q)\Delta u + (q_u + \gamma p)\Delta v + \cdots, \\ w_2 = (\beta\gamma + \theta_{uv})\Delta u + \chi\Delta v + \cdots, \\ w_3 = \pi\Delta u + (\beta\gamma + \theta_{uv})\Delta v + \cdots, \\ w_4 = 1 + \theta_u\Delta u + \theta_v\Delta v + \cdots. \end{cases}$$

If a point has coordinates $x_1, \cdots, x_4$ referred to the original tetrahedron x, x_u, x_v, x_{uv} and coordinates $\bar{x}_1, \cdots, \bar{x}_4$ referred to the neighboring tetrahedron $\bar{x}$, $\bar{x}_u$, $\bar{x}_v$, $\bar{x}_{uv}$, the transformation between the tetrahedrons is found to be given by

$$(14) \quad \begin{cases} \bar{x}_1 = x_1 - x_2 p\Delta u - x_3 q\Delta v - x_4\big[(p_v + \beta q)\Delta u \\ \qquad + (q_u + \gamma p)\Delta v\big] + \cdots, \\ \bar{x}_2 = - x_1\Delta u + x_2(1 - \theta_u\Delta u) - x_3\gamma\Delta v \\ \qquad - x_4\big[(\beta\gamma + \theta_{uv})\Delta u - \chi\Delta v\big] + \cdots, \\ \bar{x}_3 = - x_1\Delta v - x_2\beta\Delta u + x_3(1 - \theta_v\Delta v) \\ \qquad - x_4\big[\pi\Delta u + (\beta\gamma + \theta_{uv})\Delta v\big] + \cdots, \\ \bar{x}_4 = - x_2\Delta v - x_3\Delta u + x_4(1 - \theta_u\Delta u - \theta_v\Delta v) + \cdots. \end{cases}$$

Holding $v = \text{const.}$ and taking the limit in these formulas as Δu approaches zero (having first transposed certain terms and then divided by Δu), and then interchanging the roles of u and v and repeating the process, we obtain the desired *formulas for the differentiation of the local coordinates*:

$$(15) \quad \begin{cases} x_{1u} = - p x_2 - (p_v + \beta q) x_4, \\ x_{2u} = - x_1 - \theta_u x_2 - (\beta\gamma + \theta_{uv}) x_4, \\ x_{3u} = - \beta x_2 - \pi x_4, \\ x_{4u} = - x_3 - \theta_u x_4, \\ x_{1v} = - q x_3 - (q_u + \gamma p) x_4, \\ x_{2v} = - \gamma x_3 - \chi x_4, \\ x_{3v} = - x_1 - \theta_v x_3 - (\beta\gamma + \theta_{uv}) x_4, \\ x_{4v} = - x_2 - \theta_v x_4. \end{cases}$$

4. *Applications to the Local Tetrahedron.* It follows from
statements in §2 that the loci of the points x_u, x_v, x_{uv} are three
surfaces covariant to S. Moreover these surfaces and S
have their points in correspondence, corresponding points
being those that have the same curvilinear coordinates.
Without giving details of the calculations we shall simply
remark that if we place $x_1 = x$, $x_2 = x_u$, $x_3 = x_v$, $x_4 = x_{uv}$, $u^1 = u$,
$u^2 = v$, it is easy to determine a system of linear differential
equations* of the form

$$(16) \qquad \frac{\partial x_i}{\partial u^p} = \sum_{j=1}^{4} a_{ijp} x_j, \quad (i = 1, \cdots, 4 ; p = 1,2),$$

satisfied by the coordinates of these four points. Then the
results of D. Sun on quadruples of surfaces with points
in correspondence can be applied to these four surfaces.
Moreover, each of the three pairs of opposite edges of the
local tetrahedron generates two congruences with generators
in correspondence, to which the results of A. Cook on pairs
of congruences with generators in correspondence can be
applied.†

The envelopes of the faces of the local tetrahedron are
again a quadruple of surfaces with points in correspondence.
It is geometrically obvious, and easily demonstrated by (15),
that the face $x_4 = 0$ envelops S itself. *The points of contact
of the faces $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ with their envelopes are found
by* (15) *to have respectively the coordinates*

$$(17) \qquad \begin{cases} [0, -q(p_v + \beta q), -p(q_u + \gamma p), pq], \\ [-\gamma(\beta\gamma + \theta_{uv}), 0, -\chi, \gamma], \\ [-\beta(\beta\gamma + \theta_{uv}), -\pi, 0, \beta]. \end{cases}$$

If, for instance, the last two of these points are joined we
obtain a generator of a congruence covariant to S, and so a

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* Lane, *The projective differential geometry of systems of linear homo-
geneous differential equations of the first order*, Transactions of this Society,
vol. 30 (1928), pp. 785-796. See §5.

† Sun, Cook; Chicago Doctoral Dissertations (1928-29).

wide variety of possibilities for the construction of covariant configurations is suggested.

5. *Applications to the Canonical Surfaces.* If $\psi = 0$, the canonical point coincides with the point x_v. Let us suppose $\psi \neq 0$, and place

$$(18) \qquad\qquad \lambda = \phi/\psi .$$

Then the local coordinates of the canonical point are $(0, 1, -\lambda, 0)$ and the equation of the canonical plane is

$$(19) \qquad\qquad \lambda x_2 - x_3 = 0 .$$

Applying (15) to (19) we find that *the envelope of the canonical plane, as u, v vary, is a surface, which we shall call the first canonical surface S_1, and whose point of contact with the plane is given by the formulas*

$$(20) \quad \begin{cases} x_1 = -(\beta\gamma + \theta_{uv})x_4 + \pi(\lambda_v + \theta_v\lambda - \gamma\lambda^2) \\ \qquad + \chi\lambda(\lambda_u + \beta - \theta_u\lambda), \\ x_2 = -\pi + \chi\lambda^2, \quad x_3 = \lambda x_2, \\ x_4 = \lambda_u + \lambda\lambda_v + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3. \end{cases}$$

The locus of the canonical point, as u, v vary, is the second canonical surface S_2, whose tangent plane can be shown by means of (8) to have coordinates given by

$$(21) \quad \begin{cases} u_1 = \lambda_u + \lambda\lambda_v - \beta - \theta_u\lambda + \theta_v\lambda^2 + \gamma\lambda^3, \\ u_2 = \lambda u_3, \quad u_3 = p - q\lambda^2, \\ u_4 = p(\lambda_v + \theta_v\lambda + \gamma\lambda^2) + q\lambda(\lambda_u - \beta - \theta_u\lambda). \end{cases}$$

It will be observed that *the two surfaces S_1, S_2 correspond in the dualistic transformation that converts each point P of S into the tangent plane of S at P.* Analytically this kind of duality* is represented in the present notation by the transformation

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* Lane, *The correspondence between the tangent plane of a surface and its point of contact*, American Journal of Mathematics, vol. 48 (1926), p. 204.

(22) $x_1 = -(\beta\gamma + \theta_{uv})u_1 + u_4, \; x_2 = -u_3, \; x_3 = -u_2, \; x_4 = u_1$

accompanied by the substitution

(23)
$$\begin{pmatrix} \beta & \gamma & p & q & dv/du \\ -\beta & -\gamma & \pi & \chi & -dv/du \end{pmatrix}.$$

We shall make a few remarks about S_2, understanding that similar remarks, mutatis mutandis, can be made about S_1.

The tangent plane of S_2 passes through P_x in case

(24) $\lambda_u + \lambda\lambda_v - \beta - \theta_u\lambda + \theta_v\lambda^2 + \gamma\lambda^3 = 0,$

and then its line of intersection with the tangent plane of S is the second canonical tangent joining P_x to the canonical point. In this case the canonical curves of the first kind $dv - \lambda du = 0$ are dual union curves of the reciprocal of the projective normal congruence, so that the ray-point of the first canonical curve through P_x lies on the reciprocal $x_4 = x_1 = 0$ of the projective normal, being in fact the canonical point itself.

The tangent plane of S_2 meets the tangent plane of S in the reciprocal of the projective normal in case

(25) $p - q\lambda^2 = 0,$

being in this case the plane $x_1 = 0$. Then the canonical curves

(26) $dv^2 - \lambda^2 du^2$

correspond to the developables of the reciprocal of the projective normal congruence, which is therefore harmonic to the conjugate net of canonical curves.

The locus of the canonical point is a curve in case (24) and (25) are satisfied simultaneously.

6. *The Canonical Pencil of Conjugate Nets.* A pencil of conjugate nets on a surface is the class of ∞^1 conjugate nets every one of which has the property that at every point of the surface its two tangents form with the tangents of a fundamental conjugate net the same cross ratio. *The covariant pencil of conjugate nets determined by the canonical*

conjugate net (26) *will be called the canonical pencil.* The differential equation of a general net of this pencil is

(27) $dv^2 - \lambda^2 h^2 du^2 = 0$ ($h =$ const.).

The envelope of the osculating planes at P of all curves of the canonical pencil that pass through P is the axis-plane* cone

(28) $u_1 = u_2 u_3 m - \beta u_3^3 - \gamma u_2^3 = 0$

where

(29) $m = 2u_4 - (\theta_u - \lambda_u/\lambda)u_3 - (\theta_v + \lambda_v/\lambda)u_2.$

The cusp-axis of this cone joins P_x to the point

$$(0, \ - \theta_v - \lambda_v/\lambda, \ - \theta_u + \lambda_u/\lambda, 2),$$

which lies in the canonical plane in case

(30) $\lambda(\theta_v + \lambda_v/\lambda) = \theta_u - \lambda_u/\lambda.$

Let us suppose that (30) is not satisfied. Then *the plane* $\lambda_1 x_2 - x_3 = 0$, *where*

(31) $\lambda_1 = \dfrac{\theta_u - \lambda_u/\lambda}{\theta_v + \lambda_v/\lambda}$

contains the cusp-axis and the projective normal. The curves enveloped by the line of intersection of this plane with the tangent plane of S, together with their conjugates, form a new covariant conjugate net

(32) $dv^2 - \lambda_1^2 du^2 = 0,$

and the process can be repeated indefinitely. In this way we obtain a sequence of covariant conjugate nets.

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* Lane, *A general theory of conjugate nets*, Transactions of this Society, vol. 23 (1922), p. 283.