# SOME PHILOSOPHICAL ASPECTS OF MATHEMATICS*

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> "The older I get," observed Mr. van Koppen, "the more I realize that everything depends upon what a man postulates. The rest is plain sailing." (Norman Douglas, South Wind, p. 441.)

It will be well for me to begin this paper with the remark which would be superfluous later on, that it does not presume to make a contribution to mathematics. It is about mathematics, not of them. That I have nevertheless chosen it for presentation in response to the invitation of the program committee, is a result of my conviction that it is useful to reflect from time to time on the character of the structure which is being developed by an ever increasing group of workers. For, while many parts of this structure have become familiar through long acquaintance, it is steadily reaching out in new directions, opening up vistas which stimulate the imagination to envisaging still further extensions and also such as give hitherto unsuspected views of the familiar parts. Ever greater become the distances which separate those who work at different sections of this structure and it is becoming increasingly difficult for them to keep feeling with the fundamental plan which determines its development. The day is long past when unity could be secured through the coordinating agency of a single mind, or even of a group of closely associated minds; and no one can dream of holding together the lines of communication which connect the different parts of the magnificent structure. Perhaps this state of affairs, well recognized by every one, justifies an occasional speculation on the intrinsic character

* Address presented at the request of the American Mathematical Society at the joint meeting with the Mathematical Association of America and Section A of the A. A. A. S. at Nashville on December 29, 1927.

of our science. It is to indicate their speculative character that the following remarks are collected under the present title.

Two contrasting circumstances, the one old, the other of recent origin, have been the starting point for these considerations. On the one hand, we find a persistent faith, particularly on the part of those who are not very intimately familiar with the subject, in the incontestable validity of the conclusions of mathematics. They find in it an escape from the uncertainties of a doubting world, a firm rock to which to hold on in the midst of a fluid universe. To find something which is "mathematically certain" is still, as it ever was, the desire of every seeker after "truth." It is this persisting faith which gives to the mathematician a rather unique place, which provides his subject with a characterization, apparently sufficient for the uninitiated. At the same time it should lead him to a searching through of his science, to inquire whence this faith comes, whether and why it can be accepted also by those who are initiated and left undisturbed in the minds of others.*

For, on the other hand, there has grown up among mathematicians during the last decade or two, a tendency which seems to cast doubt on a large and important part of mathematics. By some this tendency has been characterized as revolutionary, as "bolshevistic" (oh, misery of words!), as subversive of the wholesomeness which had always characterized mathematics. This tendency, inaugurated by the Dutch mathematician, L. E. J. Brouwer, has several distinct aspects. It has been discussed elsewhere† and has been referred to by Professor Pierpont in the address made at yesterday's meeting. I have but to recall to you Hilbert's animadversions upon this work,‡ to have you

* A more extended discussion of "mathematical certainty" forms the subject of a paper to be published in an early number of Scientia.

† See the author's article, *Brouwer's contributions to the foundations of mathematics*, this Bulletin, vol. 30 (1924), p. 31.

‡ See Pierpont, *Mathematical rigor, past and present*, this Bulletin, vol. 34 (1928), p. 52.

realize how serious an attack upon the citadel of our science it is held to constitute. If this is indeed the character of the position of Brouwer, it is necessary for mathematicians either to refute that position decisively, or else to abandon the part of the field that is under attack and to retire to "previously prepared positions." It is the former of these alternatives which was undertaken in a paper by Barzin and Errera, at least with reference to one important aspect of Brouwer's position,* namely, his denial of the unlimited validity for mathematical logic of the Law of the Excluded Middle. And it will also be with this aspect of Brouwer's position that we shall be concerned in this paper.†

It is well known that the Aristotelian logic which ordinarily is tacitly assumed as the basis for logical reasoning proceeds from three fundamental canons, namely, the Law of Identity, the Law of Contradiction, and the Law of the Excluded

* See M. Barzin et A. Errera, *Sur la logique de M. Brouwer*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, January, 1927, p. 56.

† Although as a matter of convenience, this point of view is here ascribed to Brouwer, who is indeed its first contemporary representative and, with some of his pupils, its principal exponent, it is not his alone. For arguments on his side of the question, see H. Weyl, *Über die neue Grundlagenkrise der Mathematik*, Mathematische Zeitschrift, vol. 10 (1921), p. 39; Weyl, *Der circulus vitiosus in der heutigen Begründung der Analysis*, Jahresbericht der Vereinigung, vol. 28 (1919), p. 85; Weyl, *Randbemerkungen zu Hauptproblemen der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), (particularly p. 146–150); R. Wavre, *Y a-t-il une crise des mathématiques*, Revue de Métaphysique et de Morale, vol. 31 (1924), p. 435; also the list of references given on the last page of the article by Barzin and Errera quoted above. There is also a considerable literature on the opposing view; see J. v. Neumann, *Zur Hilbert'schen Beweistheorie*, Mathematische Zeitschrift, vol. 26 (1927), p. 1, where other important references are given, and D. Hilbert, *Über das Unendliche*, Mathematische Annalen, vol. 95 (1925), p. 161; Hilbert, *Über das Unendliche*, Jahresbericht der Vereinigung, vol. 36 (1927), p. 201. Of interest for their general bearing upon these questions are also the following papers: Cipolla, *Sue fondamenti logici della matematica secondo le recenti vedute di Hilbert*, Annali di Mathematica, (4), vol. 1 (1923), p. 19; B. Levi, *Sui procedimenti transfiniti*, Mathematische Annalen, vol. 90 (1923), p. 164; A. Müller, *Über Zahlen und Zeichen*, Mathematische Annalen, vol. 90 (1923), p. 153; M. Pasch, *Betrachtungen zur Begründung der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), p. 231; J. v. Neumann, *Eine Axiomatisierung der Mengenlehre*, Journal für Mathematik, vol. 154 (1925), p. 219.

Middle. In the forms in which they usually appear in textbooks on logic, they state respectively that (1) "$A$ is $A$"; (2) "$A$ is $B$" and "$A$ is not $B$" cannot hold simultaneously; and (3) of the two propositions "$A$ is $B$" and "$A$ is not $B$," one must always hold, whatever $A$ and $B$ may be. Brouwer's departure consists in this, that he does not accept the L.E.M. as a valid basis for all mathematical reasoning. I shall not here enter upon the reasons for his lack of confidence in the L.E.M.; but I shall refer to the logical basis which leaves it out of account as non-Aristotelian. This designation, apart from obvious reasons, is selected in order to point to an analogy with non-euclidean geometry, to which, although it is obvious, attention does not seem to have been called.* At the same time I must point out one important difference between the non-euclidean geometries and this non-Aristotelian logic. For, whereas in the former, Euclid's parallel postulate is replaced by another postulate, this logic merely omits the L.E.M. from the fundamental canons of logic. And it is in this respect that Barzin and Errera do not, in my judgment, interpret Brouwer's position correctly. For, in order to prove that this position must inevitably lead to a contradiction, they replace the L.E.M. by a new canon, namely that of the excluded fourth. Now I would be the last one to deny them the right to do this; but in so doing they create a non-Aristotelian logic different from that of Brouwer, so that their conclusion can not, in itself, be taken as a refutation of Brouwer's position. It must indeed be clear that any attempt to show that Brouwer's position which uses only two of the Aristotelian canons, leads to a contradiction, would, if successful, not leave any hope for the consistency of the classical logic. It is not likely therefore that the position can be refuted by this method.

For the purpose of my discussion, I shall make an arbitrary

* When we take cognizance of the controversial literature which has grown up in this field, we get a vivid realization of the reasons which may have led Gauss to withhold his discovery of non-euclidean geometry from publication.

division of the content of the mathematical sciences. We shall distinguish between pure mathematics (the word is to be understood as a plural here) and the application of mathematics, the latter to be taken in a sense rather different from the usual one, in which the line of demarcation is rarely definitely fixed. As a pure mathematic I designate any structure which proceeds from a set of primitive ideas, arbitrarily selected and named, and a set of primitive propositions concerning these ideas (assumptions or postulates), arbitrarily laid down, and which develops combinations of these ideas and propositions without the intervention of extraneous concepts or assumptions; the method of development is thus in an essential sense a constructive process. It is of course clear that the initiative for the making of these combinations, particularly of significant and fruitful combinations, does not lie in the assumptions and primitive ideas, but must arise out of a "capacity of the human mind," without which we could not get a step away from the axiomatic basis. The recognition of the very significant intervention of the human mind is an important element in our point of view. How it operates and why is not a part of our subject, in spite of its title; this must remain a subject for psychological investigation. An application of a mathematic is made whenever a one-to-one correspondence is set up between the primitive ideas of the mathematic and entities which have in some way or other secured objective existence, in such manner that the primitive propositions receive objective verification when these entities are put in place of the primitive ideas entering into the primitive propositions. It is at this point that a peculiarity of my division of the mathematical content into pure mathematics and applications of mathematics makes itself felt. For the entities, through the introduction of which an application of one mathematic is made, may be the primitive ideas of another mathematic, and the objective existence which is attributed to them is of a different order from that which is attached to chairs and tables. If we call such objective existence as is enjoyed by entities which, to speak without making an attempt at great refinement, can be apprehended by the senses, as being of order zero, we may say that primitive ideas which enter into a mathematic of which an application can be made through the introduction of such entities of zero order, have themselves objective existence of order one. Proceeding in this way we introduce objective existence of increasing orders; the higher the order of objective existence of an entity, the farther it is removed from what is ordinarily designated by the term. It is of course clear furthermore, that the designation of an order of objective existence attached to an entity is not a unique process, but depends upon the constructive procedure by means of which it has been reached. It is of secondary importance for our present purpose whether minimum order of objective existence greater than or equal to 1 can be attached in a unique way to every entity with which mathematicians deal, although this appears to be an interesting question for further study. In the same manner, we can assign an order to propositions, calling facts which are directly verifiable in the world of the senses, propositions of order zero; propositions which are verified by means of them, propositions of order one, and so forth. If, for instance, we set up a mathematic consisting of primitive propositions stated in terms of elements $a$ and operations $(+)$ and $(\times)$; and if these primitive propositions are verified by facts that are directly observable when we substitute, say, apples or pebbles for the elements $a$ and some concrete operations for $(+)$ and $(\times)$, then these elements $a$ and the operations $(+)$ and $(\times)$ obtain objective existence of order 1, and the primitive propositions, together with the consequences drawn from them, become propositions of order 1. If now we set up a new mathematic in terms of elements $b$ and operations $((+))$ and $((\times))$, whose primitive propositions reduce to propositions of order 1, when, for example, pairs of elements $a$ and the operations $(+)$ and $(\times)$ are put in place of the elements $b$ and the operations $((+))$ and $((\times))$, respectively, then the elements $b$ and the operations $((+))$ and $((\times))$ obtain objective existence of order 2; and so forth. The situation I am describing is somewhat obscured in many of the examples of postulational theories which are known in the literature; for in those cases, we usually set up postulates for a particular theory, that is to say, we construct a set of postulates that is to be applicable to a mathematic which has secured citizenship not by coming into the country through an immigration office, but by means of settlement before immigration laws were put into effect, or perhaps by having been smuggled in. For instance, when a set of postulates for positive integers in terms of elements $a$ and operators $(+)$ and $(\times)$ is applied, we usually make appeal not to entities and propositions for which an order of objective existence has been definitely established, but to concepts with which we are well acquainted, to neighbors with whom we are on intimate terms, even though we have never seen their naturalization certificate, indeed though we have never inquired whether they had one. But if we were to proceed on a strictly legal basis the testimony of such citizens could not be admitted; we would have to depend on original settlers or on such as possessed definitely traceable citizenship. That is to say, application of a set of postulates would then be made only through verification in terms of entities for which an order of objective existence can be established, and, perhaps preferably, in terms of entities of objective existence of order zero.

This digression on the peculiar character of the division which we have introduced into the content of mathematics seemed desirable in order to avoid misunderstanding; for our main purpose, the only point of importance is that application of a mathematic must ultimately depend upon the intervention as sponsors for our primitive ideas of entities which have, speaking roughly, the objective existence of chairs and tables, that is, what we have called objective existence of order zero, and that the verification of primitive propositions must ultimately depend upon propositions of order zero. We can sum up this conclusion in the statement that the applicability of a mathematic must ultimately depend upon the possibility of setting it into one-to-one correspondence with a part of the content of human experience.

Returning now to the consideration of a pure mathematic, it must be clear that there is no immediate meaning in the question whether it is true.* The only sense in which such a question can be made significant is by interpreting it as an inquiry as to the applicability of the mathematic, in the sense in which this term has just been explained. Each one of us can build up a mathematic, if only we are sufficiently inventive to originate a set of primitive ideas and primitive propositions. Whether these mathematics will be fruitful of results, whether they will command the abiding interest not only of their inventors but also of others, will depend upon the answers to two questions, namely whether they stimulate minds to exercise in a lasting manner that capacity for constructive combination to which we have made reference before, and whether they are applicable to significant parts of the content of human experience. I am very doubtful in my own mind whether a mathematic can have the former of these two characteristics without having the latter; but this question I must also leave aside. Certainly it seems to me so that if a mathematic has neither of the two, it will be completely sterile and will deservedly die an early death. If it have the latter property as well as the former, it will surely arouse the interest of the worker in applied science, of the "practical" man; if the property of applicability is possessed in an easily recognizable way, if the

* It is perhaps not without interest to quote here the following passage from a definitely non-mathematical source: "What was truth, after all? It was a very sensible question of Pilate's. Perhaps there *was* no truth, nor falsehood either, in any actual set of words arranged in a certain order. Perhaps they were only a neutral surface over which truth or falsehood could be cast by different minds or tongues, as a blue or an amber light is projected by turns on a colourless piece of stage canvas." (C. E. Montague, *Right Off The Map*, p. 183.)

mathematic can, through its applications, penetrate deeply into human experience, it may appeal even to the man in the street. It is, however, on account of the former property primarily that a mathematic will command the attention of mathematicians as such, because it offers them an opportunity for the investigative ability which is peculiarly theirs, namely the initiating of combinations of relatively simple elements which are rich in possibilities for further development. Of especial significance for the mathematician will be the comparison of different mathematics which are related to each other. Such relations may arise through the partial identity of their fundamental content. If two mathematics have their primitive ideas and some of their primitive propositions in common (or if there is a simple isomorphism between their primitive ideas and some of their primitive propositions), the development of either illuminates in a striking way the structure of the other. Indeed it seems to me so, that full insight in a mathematic can be gained only by comparing it with several of its related mathematics. Every one will realize that this statement is little more than a generalization by extension of the marvelous enrichment of geometrical concepts to which the non-euclidean geometries have given rise, and of the enlargement of our ideas concerning geometry and physics which the theory of relativity has brought about.

We are now in a position to view in what seems to me to be its true light the second of the circumstances mentioned at the beginning of this paper. Whether or not one agrees with the opinion according to which logic is a part of mathematics, there will be little opposition to the thesis that logic plays a fundamental role in every mathematic. And this is so because the development of a mathematic, the constructive process which we have recognized as the driving force of this development, proceeds according to the laws of logic. Indeed in every mathematic, or at least in nearly every mathematic, that has been developed thus far, it is understood, either explicitly or tacitly, that the development should remain in accord with the canons of Aristotelian logic. But, whether Aristotelian or non-Aristotelian, it must be clear that a logic must be the inherent part of every mathematic, that the primitive ideas and primitive propositions of some logic are a substructure of every mathematic. It follows therefore that every mathematic may be looked upon as involving an application of some logic, because the canons and primitive ideas of a logic must be considered as having obtained in it some sort of objective existence; they have to be appealed to if the mathematic is ever to develop beyond the embryonic stage. This carries with it, of course, that at least in so far as its structure is concerned the logic must, in some way, be looked upon as a mathematic; but for its development we can evidently not rely upon another logic, for this would clearly involve us in an infinite regress. Indeed the logic must be thought of as a part of what I have referred to above* as "the capacity of the human mind"; and the development of the logic has to proceed by the use of the residue of this capacity, left after the separating off of the logic. This residue we shall call the "bare capacity of the human mind."†

It has been recognized that the logic which underlies a mathematic has a very important bearing upon its development; and for this reason, writers, such as Frege, Peano, Whitehead and Russell have set up a pure logic as the preliminary for any mathematic. But this has been done only for the logic of classes, propositions and relations, and has not affected the fundamental canons. The significance of the departure which Brouwer has made lies in this, that he has brought the canons of logic explicitly into the foreground. By suggesting the possibility of modifying these canons as the fundamental basis even of the logic of mathematics, he

* See page 442.

† The significance of the explicit formulation of the Aristotelian canons lies in the fact that they constituted, as far as I am aware, the first attempt at an analysis of the "capacity of the human mind." Let us not fall into the error of thinking that they are a complete formulation of that capacity

has called attention to the fact that they also are a part of this basis; that what we have called the "capacity of the human mind" need not include all the Aristotelian canons. Thus he has freed the mind from the compulsory use of the Aristotelian base. The significant question to ask concerning this new logic which Brouwer builds on is not whether it is true, but whether it is applicable; and this question, in the light of our earlier discussion, means whether it is possible to set its primitive ideas and propositions into one-to-one correspondence with entities that have objective existence of order zero in human experience, that is to say in short, whether it is possible for the human mind to operate effectively with it. If it is, then the developments of this logic will aid greatly in giving insight into the capacities and possibilities of the mind, just as the comparison of two closely related postulate systems for one field of mathematics enrich our knowledge of that field. And there are indications that it is indeed possible to do so in the constructive work that has already been done by Brouwer,* Heyting† and others. The developments of a mathematic by means of this non-Aristotelian logic (usually called intuitionist mathematics) should then be looked upon in the same way as we look upon non-euclidean geometry, as an enrichment of our understanding of the way in which the human mind operates. For myself, I am very doubtful whether I could trust myself not to use the L.E.M.; but I am ready to admit that any one who can, may do important work in that way. I would go farther and open the way for using various types of logic. Each one of us can choose to work with one logic to-day and with another to-morrow, just as one's work in non-euclidean geometry does not exclude him forever from working in euclidean geometry, or as dealing with non-commutative algebras one year need not incapacitate one for occupying

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* See the list of references given by Barzin and Errera in the article quoted on p. 440.

† See *Zur intuitionistischen Axiomatik der projektiven Geometrie*, Mathematische Annalen, vol. 98 (1927), p. 491.

himself with commutative algebras the next. Whether or not his work will be fruitful is difficult to decide in advance.

If this were all there is to the position which I am taking, it would probably not be objected to seriously. For it gives greater liberty, leaving open not merely the primitive ideas and primitive propositions of a mathematic to choice, but also the rules which are to govern its development. But there is another aspect to this story, which we must now consider briefly. For the freedom which we gain is obtained at the sacrifice of a certain permanence of existence which mathematicians have usually wanted to attribute to their science, an existence in essence which is independent of any mind, and independent of time, which requires discovery to have it brought down to the realm of human apprehension. Only recently I came across the following passage in a letter from Hermite to Konigsberger, in which this idea finds expression. He wrote: "I add that these notions of analysis have their existence apart from us, that they constitute a whole of which only a part is revealed to us, incontestably although mysteriously associated with that other totality of things which we perceive by way of the senses."* To this belief, which is probably shared by most mathematicians to-day, the position which I have developed leaves little support. For if the entire mathematical structure, not the basis only, but also the guiding principles for its development, is at the choice of the individual, we have to admit that a mathematic exists only in the minds of the individual, and that without the activity of the human mind there would be no mathematics. This carries with it furthermore, that a mathematic exists only in so far as it has been developed, that is, is invented rather than discovered; that not only with regard to primitive ideas and primitive propositions, but also with regard to the logical processes, the question of truth is irrelevant; that mathematical entities may exist

* See W. Birkemeier, *Über den Bildungswert der Mathematik*, Teubner, 1923, p. 26.

to-morrow which do not exist to-day, that their existence depends upon the construction of a process, which can call them into being. It seems to me that Brouwer's position inescapably commits us to this conclusion; it clarifies points in his work, which otherwise present difficulties which it seems impossible to overcome. By disregarding this consequence of his position in their critique of his work,* Barzin and Errera seem to me to misconceive a fundamental aspect of it. In one of his most interesting papers,† Brouwer introduces an integer $k$ which is defined as the order number of that place in the decimal development of $\pi$, at which there appears for the first time a zero, immediately followed by the sequence of digits $1, 2, \cdots, 9$. When we use the non-Aristotelian logic which omits the L.E.M., we say that we do not know whether this $k$ exists, nor whether we shall ever be able to answer the question whether it exists, and that we are consequently able to construct a real number of which we do not know whether it is less than, equal to or greater than 0. If, on the other hand, we use the Aristotelian logic, we can say, using the L.E.M. that either this number $k$ exists or else it does not exist, independently of what we know about it, and that the real number determined by the use of this $k$ is either less than, equal to or greater than 0, whether or not we know which of these alternatives holds. That is to say we hold that the mathematical fact has existence, independently of whether we have discovered it.

There is therefore a fundamental difference between the characters of mathematical existence which the two points of view involve. But is this cause for alarm? I think not; in diversity there may lie strength. Indeed the new point of view adds to our insight. For when we learn that, without the use of the L.E.M. the system of real numbers is not ordered, we have increased our knowledge, just as when

* See Barzin and Errera, loc. cit., p. 59.

† See Brouwer, *Über die Bedeutung des principium tertii exclusi in der Mathematik, besonders in der Funktionentheorie*, Journal für Mathematik, vol. 154 (1925), p. 1.

we learn that in matrix algebra the product of two factors can vanish, while neither factor vanishes. In which of the two ways we proceed is a matter of choice to be determined by each individual, not for all time necessarily, but from time to time, according to his tastes and prejudices.

We come finally to the question of the bearing of this point of view on the validity of mathematical conclusions. It would seem as if the freedom which is left to the individual to build his mathematic as he chooses would not leave much basis on which to rest a compelling faith in his conclusions. Indeed, as we have seen before, the question as to the truth of these conclusions would not have any meaning. But the non-mathematician is not interested in a mathematic, but rather in the applications of a mathematic, indeed in the application which leads back from the mathematic to the content of human experience. Now we have seen that if a mathematic is to survive beyond the embryonic stage, it must be capable of development and of application. Thus a natural selection takes place which secures for a surviving mathematic those qualities which link it up with the realm of human experience. The canons of logic by which the development takes place must have their roots in the experiences of the mind; the primitive ideas and the primitive propositions must have their ultimate connections with the objective experience of the race, they must be, as it were, the result of a process of distillation, which continues throughout the history of the race; they must embody the essence of this experience, obtained through successive abstractions from its significant elements. In this way the application of a mathematic brings forth results which are relevant in human experience and which are in harmony with its fundamental characteristics. And it is this harmony of the applications of a mathematic with fundamental aspects of his experience, which constitutes their truth for the man in the street, for the man not directly concerned with mathematics; from it he derives his faith in the validity of mathematical conclusions.

We may sum up our conclusions as follows. A mathematic may be established through the free choice of a logic, and of primitive ideas and primitive propositions; if this choice is guided by wisdom, the mathematic will be capable of development and application. The application will then, sometimes directly, but more frequently through a chain of intermediate stages, have significant bearing upon the content of human experience and furnish results which may be called true. Their truth then gives, retroactively, a sound basis for belief in the validity of the conclusions of mathematics.

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