ON THE SEPARATION OF THE PLANE BY A CONTINUUM*

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In his paper, *On the separation of the plane by irreducible continua*,† W. A. Wilson obtains the following theorem. Let F be the union of two bounded continua H_1 and H_2 having these properties: H_1 and H_2 are irreducible about $A+B$; $H_1 \cdot H_2 = A+B$ where A and B are continua and $A \cdot B = 0$; H_1 and H_2 contain subcontinua C_1 and C_2 respectively such that $\alpha = C_1 \cdot C_2 \cdot A \neq 0$, $\beta = C_1 \cdot C_2 \cdot B \neq 0$, C_1 and C_2 are irreducible between α and β and $F = C_1 + C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.

In the present paper I will establish two theorems which together yield more information than Wilson's theorem.

THEOREM 1. *If A and B are two mutually exclusive continua and the bounded continua H_1 and H_2 are both irreducible about*‡ $A+B$, *and* $H_1 \cdot H_2 = A+B$, *then* $H_1 + H_2$ *is not the boundary of more than two domains.*

PROOF. Let F denote the point set $H_1 + H_2$. With the aid of the fact that $H_1 - (A+B)$ and $H_2 - (A+B)$ are§ connected, it may be easily seen that there exists an inversion of the plane about some point of the complement of F such that if, for each point set M, $\overline{M}$ denotes the image of M under this inversion, then no bounded complementary domain of $\overline{A}$ or $\overline{B}$ contains a point of $\overline{F}$. If, to each of the continua $\overline{A}$ and $\overline{B}$, all its bounded complementary domains are added and

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† This Bulletin, vol. 33 (1927), pp. 733–744.

‡ If A is a closed subset of a continuum C and no proper subcontinuum of C contains A then C is said to be irreducible about A. See W. A. Wilson, loc. cit., Definition I.

§ See W. A. Wilson, loc. cit., Lemma I.

the continuum so obtained is regarded as an element and each point that does not belong to A or to B is regarded as an element, the collection G of elements so obtained is an upper semi-continuous* collection. Furthermore, no element of the collection G separates the plane and together they fill up the plane. It follows that, if the elements of G are regarded as points, all theorems of the analysis situs of the plane hold true, in the space so obtained, in the sense described, in detail, in my paper *Concerning upper semi-continuous collections of continua.*† Hence, by a theorem used in a similar connection in the proof of Theorem 2, there exists a simple closed curve J of elements of G which separates $\overline{H}_1-(\overline{A}+\overline{B})$ from $\overline{H}_2-(\overline{A}+\overline{B})$. If, for each value of i $(i=1,2), t_i'$ denotes the segment of J whose extremities are the elements of G that contain $\overline{H}_1$ and $\overline{H}_2$, respectively, and D_i denotes the complementary domain of $\overline{F}$ that contains t_i, then, if $\overline{F}$ is the boundary of a domain, that domain must contain a point of t_1 or of t_2, and therefore must be identical with D_1 or with D_2.

THEOREM 2. *If α and β are two mutually exclusive closed point sets and each of the bounded continua C_1 and C_2 is irreducible from ‡ α to β, and $C_1 \cdot C_2=\alpha+\beta$, then C_1+C_2 is the boundary of at least two distinct domains.*

PROOF. Let F denote the point set C_1+C_2. With the help of the fact that $C_1-(\alpha+\beta)$ and $C_2-(\alpha+\beta)$ are§ connected

* See my paper, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proceedings of the National Academy, vol. 10 (1924), pp. 356–360.

† Transactions of this Society, vol. 27 (1925), pp. 416–428. In the first line of the statement of Theorem 24 on page 427 of this paper, the word "bounded" should be inserted between the words "closed" and "point".

‡ If K and L are two closed and mutually exclusive point sets, the continuum H is said to be irreducible from K to L if it contains at least one point of K and at least one point of L but contains no proper subcontinuum that does so. See Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24 (1922), pp. 144–162. Wilson uses the term *irreducible between* instead of *irreducible from.*

§ See Anna M. Mullikin, loc. cit.

it may be shown that there exists an inversion of the plane, about some point of the complement of F, such that, using the notation indicated in the above proof of Theorem 1, no bounded complementary domain of any maximal connected subset of $\bar{\alpha}+\bar{\beta}$ contains any point of $\bar{F}$. To each maximal connected subset X of $\bar{\alpha}$ add all the bounded complementary domains of X and call each point set so obtained an element. Let G_α denote the set of all such elements. Let G_β denote a set of elements determined in the same way from the maximal connected subsets of $\bar{\beta}$. Let G denote the upper semi-continuous collection whose elements are the elements of G_α, the elements of G_β, and the points that belong to no element of G_α or G_β. Let H_1 denote the set of elements $\bar{C}_1-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$ and let H_2 denote the set $\bar{C}_2-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$. The set of elements $G_\alpha+G_\beta$ is a totally disconnected set of elements and it consists of all elements common to the two closed, connected and bounded sets H_1 and H_2. Let Q denote a simple closed curve enclosing H_1+H_2. There exists an arc AB such that (1) A belongs to Q, (2) B belongs to $\bar{C}_1-(\bar{\alpha}+\bar{\beta})$ or to $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$ and (3) B is the only element of AB that belongs to H_1+H_2. Suppose B belongs to $\bar{C}_1-(\bar{\alpha}+\bar{\beta})$. Then $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$ lies in a bounded complementary domain of $Q+AB+H_1$. Call this domain D. There exists* a simple closed curve J of elements of G such that (1) J contains at least one element of $G_\alpha+G_\beta$ and encloses at least one element of $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$, (2) $(J+G_\alpha+G_\beta)-(G_\alpha+G_\beta)$ is a subset of D and it contains no element of $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$. With the help of the fact that $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$ is connected and that every element of $G_\alpha+G_\beta$ is a limit element of $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$ it easily follows that J contains $G_\alpha+G_\beta$ and encloses the whole of $\bar{C}_2-(\bar{\alpha}+\bar{\beta})$. Furthermore, $\bar{C}_1-(\bar{\alpha}+\bar{\beta})$ is wholly without J. It is easy to see that the curve J contains two segments t_1 and t_2 such that (1) for each i ($i=1, 2$) t_i contains no element of G_α or of G_β but the end elements of t_i belong to G_α and to G_β respectively and

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* See this Bulletin, vol. 33 (1927), p. 521, Abstract 38.

(2) t_1 and t_2 belong to different complementary domains of H_1+H_2. The segments t_1 and t_2 are point sets in the ordinary sense* and they are connected subsets of the complement (in the ordinary sense) of $\overline{F}$ and each of the sets $\bar{\alpha}$ and $\bar{\beta}$ contains at least one limit point of t_1 and at least one limit point of t_2. In the space of the ordinary points of the plane, for each i let D_i denote the complementary domain of $\overline{F}$ that contains t_i. The continuum $\overline{F}$ is the boundary both of D_1 and of D_2. For let N denote the boundary of D_1. If N does not contain $\overline{C}_2$ then $N\cdot\overline{C}_2$ contains no connected subset containing both a point of some element of G_1 and a point of some element of G_2. Hence there exists an arc A_1A_2 from a point A_1 of t_1 to a point A_2 of t_2 and lying, except for these points, wholly in the interior of J and having no point in common with $N\cdot\overline{C}_2$. Since $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ lies wholly without J it has no point in common with A_1A_2. Thus A_1A_2 has no point in common with N. But A_1 and A_2 belong to different complementary domains of N. Thus the supposition that N does not contain $\overline{C}_2$ has led to a contradiction. In a similar way it may be shown that N contains $\overline{C}_1$. Therefore $\overline{F}$ is the boundary of D_1. By a similar argument it may be shown to be the boundary of D_2. It follows that F is the boundary of at least two domains.

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* This does not imply that they are necessarily subsets of *arcs* whose elements are all points in the ordinary sense. It is possible, for example, that t_1 may be the set of all points of the graph of $y=\sin(1/x)$ that lie between the lines $x=0$ and $x=1$. This point set is not a subset of any arc whose elements are all points but it is a segment of an arc of which the only element which is not a point is the straight line interval from (0,1) to (0,−1).